(30.0) Holomorphic functions defined by integrals.— In this set of notes, we will introduce a method of constructing holomorphic functions via finite and infinite integrals. The general approach is going to be the following:

\[ \text{Input: } K(t, z) : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{C} \quad \text{Output: } f(z) = \int_{0}^{\infty} K(t, z) \, dt \]

Here, \( \Omega \subset \mathbb{C} \) is an open set. \( K(t, z) \) is a function defined on \( t \in \mathbb{R}_{\geq 0} \) and \( z \in \Omega \), which takes values in \( \mathbb{C} \). We will have to impose various conditions on \( K(t, z) \) (see §30.3 below) in order to ensure two properties:

- \( f : \Omega \to \mathbb{C} \) is a holomorphic function.
- \( f'(z_0) = \int_{0}^{\infty} \partial_z K(t, z_0) \, dt \), for every \( z_0 \in \Omega \). Here \( \partial_z K \) is the derivative of \( K(t, z) \) with respect to the second variable \( z \). More explicitly,
  \[ \partial_z K(t, z_0) = \lim_{h \to 0, h \in \mathbb{C}} \frac{K(t, z_0 + h) - K(t, z_0)}{h} . \]
  (Among other things, we will assume that this limit exists.)

Remarks.

(1) Such integrals appear very frequently in the theory of differential equations, asymptotic analysis and physics. A special case of great importance, known as Laplace transform, is given in §30.5 below.

(2) The term infinite integrals was coined by Hardy \(^1\) in 1902, to suggest the analogy between \( \int_{0}^{\infty} \) and \( \sum_{n=0}^{\infty} \). According to this philosophy, the diagram above is a continuous version of the one I sketched in Lecture 21, §21.0. And, the main theorem of these notes (§30.3) is the continuous analogue of Weierstrass’ theorem on uniform convergence (Lecture 22, §22.6). By “continuous version”, I mean: instead of summing over discrete subscript \( n = 0, 1, 2, \ldots \), we are “summing over” continuous parameter \( t \in (0, \infty) \).

(30.1) Finite case.— Let us begin by considering the finite integrals first. The set up is as follows. We are given a closed bounded interval \([a, b] \subset \mathbb{R}\), an open set \( \Omega \subset \mathbb{C} \), and a

\(^1\text{G.H. Hardy (1877-1947)}\)
function $K(t, z)$ of two variables (taking values in $\mathbb{C}$): $K : [a, b] \times \Omega \to \mathbb{C}$. Assume that this function satisfies the following hypotheses.

**Assumption 1.** $K(t, z)$ is a continuous function of $(t, z) \in [a, b] \times \Omega$.

**Assumption 2.** The continuity of $K(t, z)$, in the variable $z \in \Omega$, is uniform with respect to $t$.

**Assumption 3.** For a fixed $t_0 \in [a, b]$, the function $K(t_0, -) : \Omega \to \mathbb{C}$ is holomorphic.

**Assumption 4.** $\partial_z K(t, z)$ is a continuous function of $t$.

**Theorem.** For $K(t, z)$ satisfying Assumptions 1-4 above, let us define $f : \Omega \to \mathbb{C}$ by:

$$f(z) = \int_a^b K(t, z) \, dt$$

Then, $f$ is a holomorphic function and $f'(z_0) = \int_a^b \partial_z K(t, z_0) \, dt$.

**Remarks on the assumptions 1-4.**

1. Assumption 1 is stronger than saying that $K(t, z)$ is continuous function of $t$ and $z$ individually. You must have seen the following example in Calculus III. Let $g(x, y)$ be a real valued function of two real variables, defined by:

$$g(x, y) = \begin{cases} 
xy & \text{if } (x, y) \neq (0, 0) \\
0 & \text{if } x = y = 0
\end{cases}$$

Then, for a fixed $x_0 \in \mathbb{R}$, $g(x_0, y)$ is a continuous function of $y$ (similarly, for a fixed $y_0$, $g(x, y_0)$ is a continuous function of $x$), but it is not a continuous function of $(x, y)$ together.

2. Recall that $K(t, z)$ is continuous at $z_0 \in \Omega$ (assuming $t \in [a, b]$ is fixed) means: given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \text{ implies that } |K(t, z) - K(t, z_0)| < \epsilon.$$ 

This $\delta$, in general, will depend on $t$. Assumption 2 means that a $\delta$ can be chosen to work for all $t \in [a, b]$ simultaneously.

Assumption 2 is actually not needed at all. It follows from assumption 1 (using Heine–Borel theorem given in Optional Reading A §A.7). I have only it solely for the sake of convenience in writing the proof below.

3. Assumption 3 is absolutely crucial. It ensures that the following limit exists (for any $t \in [a, b]$ and $z_0 \in \Omega$):

$$\lim_{h \to 0 \atop h \in \mathbb{C}} \frac{K(t, z_0 + h) - K(t, z_0)}{h},$$
which we define to be \( \partial_z K(t, z_0) \). This assumption will also allow us to write (for a fixed \( t \), \( K(t, z_0) \) as a contour integral, using Cauchy’s integral formula.

(4) Now that \( \partial_z K \) is defined, assumption 4 requires it to be continuous in \( t \in [a, b] \) variable, so that the integral \( \int_a^b \partial_z K(t, z) \, dt \) exists.

(30.3) **Proof of Theorem 30.1.** – Theorem 30.2 is proved exactly as Weierstrass’ theorem on uniform convergence (Lecture 22, §22.6). We begin by proving that \( f(z) \) is continuous.

So, let \( z_0 \in \Omega \) and let \( \varepsilon > 0 \) be given. According to Assumption 2, we can find \( \delta > 0 \) such that:

\[
0 < |z - z_0| < \delta \text{ implies } |K(t, z) - K(t, z_0)| < \frac{\varepsilon}{b - a} \text{ for every } t \in [a, b].
\]

With this \( \delta \), we can conclude that, for every \( z \) such that \( 0 < |z - z_0| < \delta \), we have:

\[
|f(z) - f(z_0)| = \left| \int_a^b (K(t, z) - K(t, z_0)) \, dt \right| \leq \int_a^b |K(t, z) - K(t, z_0)| \, dt.
\]

By the important inequality (§12.7), we get

\[
|f(z) - f(z_0)| < \frac{\varepsilon}{b - a} \cdot (b - a) = \varepsilon.
\]

Hence \( f(z) \) is continuous.

Next, we will prove that \( f'(z_0) \) exists for every \( z_0 \in \Omega \). First of all, let us pick a positive real number \( r \in \mathbb{R}_{>0} \) so that the open disc \( D(z_0; r) \subset \Omega \) (exists, since \( \Omega \) is open). Choose another number \( 0 < \rho < r \) so that the counterclockwise oriented circle \( C_\rho \) of radius \( \rho \), centered at \( z_0 \) lies in \( \Omega \). By Cauchy’s integral formula, for every \( w \in D(z_0; \rho) \) (and \( t \in [a, b] \)) we have:

\[
K(t, w) = \frac{1}{2\pi i} \int_{C_\rho} \frac{K(t, z)}{z-w} \, dz.
\]

(we are using Assumption 3 - for a fixed \( t \), \( K(t, z) \) is a holomorphic function of \( z \in \Omega \)).

Now we are ready to show that the limit \( \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \) exists. We are going to assume that \( |h| < \rho \). This gives:

\[
\frac{f(z_0 + h) - f(z_0)}{h} = \frac{1}{h} \int_a^b (K(t, z_0 + h) - K(t, z_0)) \, dt
\]

\[
= \frac{1}{2\pi i h} \int_a^b \left( \int_{C_\rho} \frac{K(t, z)}{z-z_0-h} - \frac{K(t, z)}{z-z_0} \, dz \right) \, dt
\]

\[
= \frac{1}{2\pi i} \int_a^b \left( \int_{C_\rho} \frac{K(t, z)}{(z-z_0-h)(z-z_0)} \, dz \right) \, dt
\]
Now we have to prove that the limit as $h \to 0$ of the last integral exists and is given by
\[ \int_a^b \partial_z K(t, z_0) \, dt \] (which exists by Assumption 4). By Cauchy’s integral formula, we have:
\[
\partial_z K(t, z_0) = \frac{1}{2\pi i} \int_{C_\rho} \frac{K(t, z)}{(z - z_0)^2} \, dz.
\]

Now our proof proceeds as the one given in Lecture 17, §17.2. Let $M$ be the maximum of $|K(t, z)|$ as $t \in [a, b]$ and $z$ lies on $C_\rho$ (this is a compact set, and $K(t, z)$ is continuous in both variables jointly by Assumption 1 - so absolute maximum exists and is finite). We obtain the following bound (see §17.2 for details of this computation):
\[
\left| \frac{1}{2\pi i} \int_a^b \left( \int_{C_\rho} \frac{K(t, z)}{(z - z_0 - h)(z - z_0)} - \frac{K(t, z)}{(z - z_0)^2} \right) \, dz \right| dt \leq \frac{M(b - a)}{\rho(\rho - |h|)}|h|,
\]
which $\to 0$ and $h \to 0$. Hence, we can conclude that $f'(z_0)$ exists and:
\[
f'(z_0) = \int_a^b \partial_z K(t, z_0) \, dt.
\]

(30.4) Infinite integral.— Now we consider the case of the infinite integral. We have $K : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{C}$, and we are going to define $f(z) = \int_0^\infty K(t, z) \, dt$.

Recall (from Calculus II) that the infinite integral is defined as the limit of finite ones:
\[
\int_0^\infty g(t) \, dt = \lim_{R \to \infty} \int_0^R g(t) \, dt.
\]
The existence of such a limit (by Cauchy’s criterion - see Lecture 21, page 2) amounts to saying that: for every $\varepsilon > 0$, we can find $R$ such that:
\[
\text{For every } S > R \text{ we have } \left| \int_R^S g(t) \, dt \right| < \varepsilon.
\]
In particular $\left| \int_R^\infty g(t) \, dt \right| \leq \varepsilon$. Meaning, the integral remaining after $R$ can be made as small as we want. This statement is in fact equivalent to Cauchy’s criterion, because $\int_R^S = \int_\infty^\infty - \int_\infty^S$.

Assumptions on $K(t, z)$. We have to continue imposing the assumptions laid out in §30.1 above. Just to write them again:

Assumption 1. $K(t, z)$ is a continuous function of $(t, z) \in \mathbb{R}_{\geq 0} \times \Omega$.

Assumption 2. The continuity of $K(t, z)$ in $z$ variable is uniform with respect to closed and bounded intervals in $\mathbb{R}$.

Assumption 3. For fixed $t_0 \in \mathbb{R}_{\geq 0}$, $K(t_0, z)$ is a holomorphic function of $z$.

Assumption 4. $\partial_z K(t, z)$ is a continuous function of $t$. 

Additionally, we have to assume that \( \int_0^\infty K(t, z) \, dt \) exists, which amounts to the existence of a limit. We will require this limit to exist \textit{uniformly in} \( z \in \Omega \).  

**Assumption 5.** The limit \( \lim_{R \to \infty} \int_0^R K(t, z) \, dt \) exists uniformly in \( z \). To spell it out: for every compact set \( D \subset \mathbb{C} \) contained in \( \Omega \), and every \( \varepsilon > 0 \), we can find \( R \) such that 
\[
\left| \int_R^\infty K(t, z) \, dt \right| < \varepsilon \quad \text{for every} \quad z \in D.
\]

**Theorem.** Let \( K(t, z) \) be a function of two variables \( t \in \mathbb{R}_0^+ \) and \( z \in \Omega \) satisfying Assumptions 1-5 above. Define:
\[
f(z) = \int_0^\infty K(t, z) \, dt
\]
Then, \( f(z) \) is a holomorphic function on \( \Omega \) and \( f'(z) = \int_0^\infty \partial_z K(t, z) \, dt \).

**Proof.** This theorem is a consequence of Theorem 30.1 and Weierstrass’ theorem on uniform convergence (Lecture 22, §22.6). In a bit more detail, for every \( N \in \mathbb{Z}_{\geq 1} \), define \( f_N(z) = \int_0^N K(t, z) \, dt \). This is a holomorphic function by Theorem 30.1 and the sequence of functions \( \{f_N(z)\}_{N=1}^\infty \) converges uniformly by Assumption 5. So, the limit is a holomorphic function by Weierstrass’ theorem, and the derivative is the uniform limit of \( \{f'_N(z)\} \). The theorem is proved. \( \square \)

(30.5) Laplace transform.-- Let \( \varphi(t) \) be a continuous function of a real variable \( t \), taking values in \( \mathbb{C} \). The \textit{Laplace transform} of \( \varphi \), denoted by \( \mathcal{L}\varphi(z) \), is defined as:
\[
\mathcal{L}\varphi(z) = \int_0^{\infty} \varphi(t)e^{-zt} \, dt
\]

Let us assume that there exists constants (independent of \( t \)), \( r, C \in \mathbb{R}_0^+ \) such that 
\[
|\varphi(t)| < Ce^{rt} \quad \text{for every} \quad t \in \mathbb{R}_0^+.
\]

**Claim.** \( \mathcal{L}\varphi \) is a holomorphic function on \( \Omega = \{ z \in \mathbb{C} : \text{Re}(z) > r \} \).

**Proof.** In order to use Theorem 30.4, we have to verify Assumptions 1-5 for \( K(t, z) = \varphi(t)e^{-zt} : \mathbb{R}_0^+ \times \Omega \to \mathbb{C} \).

Assumptions 1-4 hold without any difficulty. The only non-trivial thing to check is Assumption 5: \( \lim_{R \to \infty} \int_0^R K(t, z) \, dt \) exists, uniformly in \( z \in \Omega \).

\[^3\text{Recall that we agreed: “uniformly” without any other specifications means “with respect to compact sets }D \subset \mathbb{C} \text{ contained in } \Omega \text{”}\]
Let \( D \subset \Omega \) be a closed and bounded set. Pick \( r_1 > r \) such that \( \text{Re}(z) \geq r_1 \) for every \( z \in D \). Then, for every \( z \in D \) we have \( |e^{-zt}| = e^{-t \text{Re}(z)} \leq e^{-r_1 t} \).

\[
\left| \int_R^\infty \varphi(t)e^{-zt} \, dt \right| \leq \int_R^\infty Ce^{-(r_1-r)t} \, dt = Ce^{-(r_1-r)R} \frac{r_1}{r_1-r}.
\]

So, given \( \varepsilon > 0 \), choose \( R \) large enough so that \( e^{-(r_1-r)R} < \frac{r_1}{C} \varepsilon \) (this can be done since \( \lim_{R \to \infty} e^{-(r_1-r)R} = 0 \)). Then, for every \( z \in D \), we have that \( \left| \int_R^\infty \varphi(t)e^{-zt} \, dt \right| < \varepsilon \). The claim is proved. \( \square \)

Examples.

1. \( \varphi(t) = 1 \). We get \( L\varphi(z) = \int_0^\infty e^{-zt} \, dt = -\frac{1}{z} \left[ e^{-zt} \right]_t^\infty \). Assuming \( \text{Re}(z) > 0 \), we can conclude that \( \lim_{t \to \infty} e^{-zt} = 0 \). Thus \( L\varphi(z) = \frac{1}{z} \) for \( \text{Re}(z) > 0 \).

2. \( \varphi(t) = t \). Again we can compute (integration by parts):

\[
L\varphi(z) = \int_0^\infty te^{-zt} \, dt = -\frac{1}{z} \left[ te^{-zt} \right]_t^\infty + \frac{1}{z} \int_0^\infty e^{-zt} \, dt = \frac{1}{z^2},
\]

assuming \( \text{Re}(z) > 0 \). Note that the first term on the right–hand side of the equation vanishes under this assumption.

3. (Exercise.) Let \( \varphi(t) = \frac{t^n}{n!} \) (\( n \in \mathbb{Z}_{\geq 0} \)). Prove that \( L\varphi(z) = z^{-n-1} \), for \( \text{Re}(z) > 0 \).

4. (Exercise.) Let \( \varphi(t) = e^t \). Prove that \( L\varphi(z) = \frac{1}{z-1} \), for \( \text{Re}(z) > 1 \).

5. (Exercise.) Let \( \varphi(t) \) be a continuous function of a real variable \( t \). Let \( c \in \mathbb{C} \) be a constant and define \( \psi(t) = e^{ct}\varphi(t) \). Prove the following identity:

\[
L\psi(z) = L\varphi(z-c)
\]

(30.6) Gamma function.– The gamma function \( \Gamma(z) \) was defined by Euler in 1729 as the following infinite integral:

\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t} \, dt
\]

We will prove next week that this integral is well–defined for \( \text{Re}(z) > 0 \). For now, I leave you with the following exercise (which is in fact Exercise from Example (3) in §30.5 above):

**Exercise.** Let \( n \geq 0 \). Prove that \( \int_0^\infty t^n e^{-t} \, dt = n! \)

(This exercise was the motivation behind defining \( \Gamma(z) \) in terms of infinite integral as written above - \( \Gamma \) function was discovered by Euler as a (or the with some additional assumptions) solution to the problem of interpolating points \( \{(n, n!): n \in \mathbb{Z}_{\geq 0}\} \). This problem was suggested to Euler by Goldbach.)