COMPLEX ANALYSIS: LECTURE 30

(30.0) Holomorphic functions defined by integrals.– In this set of notes, we will introduce a method of constructing holomorphic functions via finite and infinite integrals. The general approach is going to be the following:

$$\boxed{\text{Input: } K(t,z) : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{C}} \rightsquigarrow \boxed{\text{Output: } f(z) = \int_0^\infty K(t,z)\, dt}$$

Here, $\Omega \subset \mathbb{C}$ is an open set. $K(t,z)$ is a function defined on $t \in \mathbb{R}_{\geq 0}$ and $z \in \Omega$, which takes values in $\mathbb{C}$. We will have to impose various conditions on $K(t,z)$ (see §30.3 below) in order to ensure two properties:

- $f : \Omega \to \mathbb{C}$ is a holomorphic function.
- $f'(z_0) = \displaystyle\int_0^\infty \partial_z K(t,z_0)\, dt$, for every $z_0 \in \Omega$. Here $\partial_z K$ is the derivative of $K(t,z)$ with respect to the second variable z. More explicitly,
$$\partial_z K(t,z_0) = \lim_{\substack{h\to 0\\ h\in\mathbb{C}}} \frac{K(t,z_0+h) - K(t,z_0)}{h}\ .$$
(Among other things, we will assume that this limit exists.)

Remarks.

(1) Such integrals appear very frequently in the theory of differential equations, asymptotic analysis and physics. A special case of great imporatance, known as Laplace transform, is given in §30.5 below.

(2) The term *infinite integrals* was coined by Hardy [1] in 1902, to suggest the analogy between $\displaystyle\int_0^\infty$ and $\displaystyle\sum_{n=0}^\infty$. According to this philosophy, the diagram above is a continuous version of the one I sketched in Lecture 21, §21.0. And, the main theorem of these notes (§30.3) is the continuous analogue of Weierstrass' theorem on uniform convergence (Lecture 22, §22.6). By "continuous version", I mean: instead of summing over discrete subscript $n = 0, 1, 2, \ldots$, we are "summing over" continuous parameter $t \in (0, \infty)$.

(30.1) Finite case.– Let us begin by considering the finite integrals first. The set up is as follows. We are given a closed bounded interval $[a,b] \subset \mathbb{R}$, an open set $\Omega \subset \mathbb{C}$, and a

[1]G.H. Hardy (1877-1947)

function $K(t,z)$ of two variables (taking values in $\mathbb{C}$): $K:[a,b]\times\Omega\to\mathbb{C}$. Assume that this function satisfies the following hypotheses.

Assumption 1. $K(t,z)$ is a contiuous function of $(t,z)\in[a,b]\times\Omega$.

Assumption 2. The continuity of $K(t,z)$, in the variable $z\in\Omega$, is uniform with respect to t.

Assumption 3. For a fixed $t_0\in[a,b]$, the function $K(t_0,-):\Omega\to\mathbb{C}$ is holomorphic.

Assumption 4. $\partial_z K(t,z)$ is a continuous function of t.

Theorem. *For $K(t,z)$ satisfying Assumptions 1-4 above, let us define $f:\Omega\to\mathbb{C}$ by:*

$$\boxed{f(z)=\int_a^b K(t,z)\,dt}$$

Then, f is a holomorphic function and $f'(z_0)=\int_a^b \partial_z K(t,z_0)\,dt$.

(30.2) Remarks on the assumptions 1-4.–

(1) Assumption 1 is stronger than saying that $K(t,z)$ is contiuous function of t and z *individually*. You must have seen the following example in Calculus III. Let $g(x,y)$ be a real valued function of two real variables, defined by:

$$g(x,y)=\begin{cases}\frac{xy}{x^2+y^2} & \text{if } (x,y)\neq(0,0)\\ 0 & \text{if } x=y=0\end{cases}$$

Then, for a fixed $x_0\in\mathbb{R}$, $g(x_0,y)$ is a continuous function of y (similarly, for a fixed y_0, $g(x,y_0)$ is a contiuous function of x), but it is not a contiuous function of (x,y) together.

(2) Recall that $K(t,z)$ is contiuous at $z_0\in\Omega$ (assuming $t\in[a,b]$ is fixed) means: given $\varepsilon>0$, there exists $\delta>0$ such that

$$0<|z-z_0|<\delta \text{ implies that } |K(t,z)-K(t,z_0)|<\epsilon.$$

This δ, in general, will depend on t. Assumption 2 means that a δ can be chosen to work for all $t\in[a,b]$ simultaneously.

Assumption 2 is actually not needed at all. It follows from assumption 1 (using Heine–Borel theorem given in Optional Reading A §A.7). I have only it solely for the sake of convenience in writing the proof below.

(3) Assumption 3 is absolutely crucial. It ensures that the following limit exists (for any $t\in[a,b]$ and $z_0\in\Omega$):

$$\lim_{\substack{h\to 0\\ h\in\mathbb{C}}}\frac{K(t,z_0+h)-K(t,z_0)}{h}\ ,$$

which we define to be $\partial_z K(t,z_0)$. This assumption will also allow us to write (for a fixed t), $K(t,z_0)$ as a contour integral, using Cauchy's integral formula.

(4) Now that $\partial_z K$ is defined, assumption 4 requires it to be contiuous in $t \in [a,b]$ variable, so that the integral $\displaystyle\int_a^b \partial_z K(t,z)\,dt$ exists.

(30.3) Proof of Theorem 30.1.– [2] Theorem 30.2 is proved exactly as Weierstrass' theorem on uniform convergence (Lecture 22, §22.6). We begin by proving that $f(z)$ is contiuous.

So, let $z_0 \in \Omega$ and let $\varepsilon > 0$ be given. According to Assumtion 2, we can find $\delta > 0$ such that:

$$0 < |z - z_0| < \delta \text{ implies } |K(t,z) - K(t,z_0)| < \frac{\varepsilon}{b-a} \text{ for every } t \in [a,b].$$

With this δ, we can conclude that, for every z such that $0 < |z - z_0| < \delta$, we have:

$$|f(z) - f(z_0)| = \left|\int_a^b (K(t,z) - K(t,z_0))\,dt\right| \le \int_a^b |K(t,z) - K(t,z_0)|\,dt$$

By the important inequality (§12.7), we get

$$|f(z) - f(z_0)| < \frac{\varepsilon}{b-a}\cdot(b-a) = \varepsilon.$$

Hence $f(z)$ is contiuous.

Next, we will prove that $f'(z_0)$ exists for every $z_0 \in \Omega$. First of all, let us pick a positive real number $r \in \mathbb{R}_{>0}$ so that the open disc $D(z_0;r) \subset \Omega$ (exists, since Ω is open). Choose another number $0 < \rho < r$ so that the counterclockwise oriented circle C_ρ of radius ρ, centered at z_0 lies in Ω. By Cauchy's integral formula, for every $w \in D(z_0;\rho)$ (and $t \in [a,b]$) we have:

$$K(t,w) = \frac{1}{2\pi\mathbf{i}}\int_{C_\rho} \frac{K(t,z)}{z-w}\,dz.$$

(we are using Assumption 3 - for a fixed t, $K(t,z)$ is a holomorphic function of $z \in \Omega$).

Now we are ready to show that the limit $\displaystyle\lim_{h\to 0}\frac{f(z_0+h)-f(z_0)}{h}$ exists. We are going to assume that $|h| < \rho$. This gives:

$$\begin{aligned}
\frac{f(z_0+h)-f(z_0)}{h} &= \frac{1}{h}\int_a^b (K(t,z_0+h) - K(t,z_0))\,dt \\
&= \frac{1}{2\pi\mathbf{i}h}\int_a^b \left(\int_{C_\rho} \frac{K(t,z)}{z-z_0-h} - \frac{K(t,z)}{z-z_0}\,dz\right) dt \\
&= \frac{1}{2\pi\mathbf{i}}\int_a^b \left(\int_{C_\rho} \frac{K(t,z)}{(z-z_0-h)(z-z_0)}\,dz\right) dt
\end{aligned}$$

[2] Optional

Now we have to prove that the limit as $h \to 0$ of the last integral exists and is given by $\displaystyle\int_a^b \partial_z K(t, z_0)\, dt$ (which exists by Assumption 4). By Cauchy's integral formula, we have:

$$\partial_z K(t, z_0) = \frac{1}{2\pi\mathbf{i}} \int_{C_\rho} \frac{K(t,z)}{(z-z_0)^2}\, dz.$$

Now our proof proceeds as the one given in Lecture 17, §17.2. Let M be the maximum of $|K(t,z)|$ as $t \in [a,b]$ and z lies on C_ρ (this is a compact set, and $K(t,z)$ is continuous in both variables jointly by Assmption 1 - so absolute maximum exists and is finite). We obtain the following bound (see §17.2 for details of this computation):

$$\left| \frac{1}{2\pi\mathbf{i}} \int_a^b \left(\int_{C_\rho} \left(\frac{K(t,z)}{(z-z_0-h)(z-z_0)} - \frac{K(t,z)}{(z-z_0)^2} \right) dz \right) dt \right| \leq \frac{M(b-a)}{\rho(\rho - |h|)} |h| \ ,$$

which $\to 0$ and $h \to 0$. Hence, we can conclude that $f'(z_0)$ exists and:

$$f'(z_0) = \int_a^b \partial_z K(t, z_0)\, dt \ .$$

(30.4) Infinite integral.– Now we consider the case of the infinite integral. We have $K : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{C}$, and we are going to define $f(z) = \displaystyle\int_0^\infty K(t,z)\, dt$.

Recall (from Calculus II) that the infinite integral is defined as the limit of finite ones: $\displaystyle\int_0^\infty g(t)\, dt = \lim_{R\to\infty} \int_0^R g(t)\, dt$. The existence of such a limit (by Cauchy's criterion - see Lecture 21, page 2) amounts to saying that: for every $\varepsilon > 0$, we can find R such that:

$$\text{For every } S > R \text{ we have } \left| \int_R^S g(t)\, dt \right| < \varepsilon.$$

In particular $\displaystyle\left| \int_R^\infty g(t)\, dt \right| \leq \varepsilon$. Meaning, the integral remaining after R can be made as small as we want. This statement is in fact equivalent to Cauchy's criterion, because $\int_R^S = \int_R^\infty - \int_S^\infty$.

Assumptions on $K(t,z)$. We have to continue imposing the assumptions laid out in §30.1 above. Just to write them again:

Assumption 1. $K(t,z)$ is a contiuous function of $(t,z) \in \mathbb{R}_{\geq 0} \times \Omega$.

Assumption 2. The continuity of $K(t,z)$ in z variable is uniform with respect to closed and bounded intervals in $\mathbb{R}$.

Assumption 3. For fixed $t_0 \in \mathbb{R}_{\geq 0}$, $K(t_0, z)$ is a holomorphic function of z.

Assumption 4. $\partial_z K(t,z)$ is a contiuous function of t.

Additionally, we have to assume that $\int_0^\infty K(t,z)\,dt$ exists, which amounts to the existence of a limit. We will require this limit to exist *uniformly in* $z \in \Omega$ [3].

Assmption 5. The limit $\lim_{R\to\infty} \int_0^R K(t,z)\,dt$ exists uniformly in z. To spell it out: for every compact set $D \subset \mathbb{C}$ contained in Ω, and every $\varepsilon > 0$, we can find R such that

$$\left| \int_R^\infty K(t,z)\,dt \right| < \varepsilon \text{ for every } z \in D.$$

Theorem. *Let $K(t,z)$ be a function of two variables $t \in \mathbb{R}_{\geq 0}$ and $z \in \Omega$ satisfying Assumptions 1-5 above. Define:*

$$\boxed{f(z) = \int_0^\infty K(t,z)\,dt}$$

Then, $f(z)$ is a holomorphic function on Ω and $f'(z) = \int_0^\infty \partial_z K(t,z)\,dt$.

Proof. This theorem is a consequence of Theorem 30.1 and Weierstrass' theorem on uniform convergence (Lecture 22, §22.6). In a bit more detail, for every $N \in \mathbb{Z}_{\geq 1}$, define $f_N(z) = \int_0^N K(t,z)\,dt$. This is a holomorphic function by Theorem 30.1 and the sequence of functions $\{f_N(z)\}_{N=1}^\infty$ converges uniformly by Assumption 5. So, the limit is a holomorphic function by Weierstrass' theorem, and the derivative is the uniform limit of $\{f_N'(z)\}$. The theorem is proved. $\square$

(30.5) Laplace transform.– Let $\varphi(t)$ be a continuous function of a real variable t, taking values in $\mathbb{C}$. The *Laplace transform* of φ, denoted by $\mathcal{L}\varphi(z)$, is defined as:

$$\boxed{\mathcal{L}\varphi(z) = \int_0^\infty \varphi(t) e^{-zt}\,dt}$$

Let us assume that there exists constants (independent of t), $r, C \in \mathbb{R}_{>0}$ such that $|\varphi(t)| < Ce^{rt}$ for every $t \in \mathbb{R}_{\geq 0}$.

Claim. $\mathcal{L}\varphi$ is a holomorphic function on $\Omega = \{z \in \mathbb{C} : \mathrm{Re}(z) > r\}$.

Proof. In order to use Theorem 30.4, we have to verify Assumptions 1-5 for $K(t,z) = \varphi(t)e^{-zt} : \mathbb{R}_{\geq 0} \times \Omega \to \mathbb{C}$.

Assumptions 1-4 hold without any difficulty. The only non–trivial thing to check is Assumption 5: $\lim_{R\to\infty} \int_0^R K(t,z)\,dt$ exists, uniformly in $z \in \Omega$.

[3] Recall that we agreed: "uniformly" without any other specifications means "with respect to compact sets $D \subset \mathbb{C}$ contained in Ω"

Let $D \subset \Omega$ be a closed and bounded set. Pick $r_1 > r$ such that $\mathrm{Re}(z) \geq r_1$ for every $z \in D$. Then, for every $z \in D$ we have $|e^{-zt}| = e^{-t\,\mathrm{Re}(z)} \leq e^{-r_1 t}$.

$$\left| \int_R^\infty \varphi(t) e^{-zt}\, dt \right| \leq \int_R^\infty C e^{-(r_1 - r)t}\, dt = C \frac{e^{-(r_1-r)R}}{r_1 - r}.$$

So, given $\varepsilon > 0$, choose R large enough so that $e^{-(r_1-r)R} < \frac{r_1-r}{C}\varepsilon$ (this can be done since $\lim\limits_{R\to\infty} e^{-(r_1-r)R} = 0$). Then, for every $z \in D$, we have that $\left| \int_R^\infty \varphi(t) e^{-zt}\, dt \right| < \varepsilon$. The claim is proved. □

Examples.

(1) $\varphi(t) = 1$. We get $\mathcal{L}\varphi(z) = \displaystyle\int_0^\infty e^{-zt}\, dt = -\frac{1}{z}\left[e^{-zt}\right]_{t=0}^\infty$. Assuming $\mathrm{Re}(z) > 0$, we can conclude that $\lim_{t\to\infty} e^{-zt} = 0$. Thus $\mathcal{L}\varphi(z) = \dfrac{1}{z}$ for $\mathrm{Re}(z) > 0$.

(2) $\varphi(t) = t$. Again we can compute (integration by parts):

$$\mathcal{L}\varphi(z) = \int_0^\infty t e^{-zt}\, dt = -\frac{1}{z}\left[t e^{-zt}\right]_{t=0}^\infty + \frac{1}{z}\int_0^\infty e^{-zt}\, dt = \frac{1}{z^2}\ ,$$

assuming $\mathrm{Re}(z) > 0$. Note that the first term on the right–hand side of the equation vanishes under this assumption.

(3) (*Exercise.*) Let $\varphi(t) = \dfrac{t^n}{n!}$ $(n \in \mathbb{Z}_{\geq 0})$. Prove that $\mathcal{L}\varphi(z) = z^{-n-1}$, for $\mathrm{Re}(z) > 0$.

(4) (*Exercise.*) Let $\varphi(t) = e^t$. Prove that $\mathcal{L}\varphi(z) = \dfrac{1}{z-1}$, for $\mathrm{Re}(z) > 1$.

(5) (*Exercise.*) Let $\varphi(t)$ be a contiuous function of a real variable t. Let $c \in \mathbb{C}$ be a constant and define $\psi(t) = e^{ct}\varphi(t)$. Prove the following identity:

$$\mathcal{L}\psi(z) = \mathcal{L}\varphi(z - c)$$

(30.6) Gamma function.– The gamma function $\Gamma(z)$ was defined by Euler in 1729 as the following infinite integral:

$$\boxed{\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt}$$

We will prove next week that this integral is well–defined for $\mathrm{Re}(z) > 0$. For now, I leave you with the following exercise (which is in fact Exercise from Example (3) in §30.5 above):

Exercise. Let $n \geq 0$. Prove that $\displaystyle\int_0^\infty t^n e^{-t}\, dt = n!$

(This exercise was the motivation behind defining $\Gamma(z)$ in terms of infinite integral as written above - Γ function was discovered by Euler as a (or *the* with some additional assumptions) solution to the problem of interpolating points $\{(n, n!) : n \in \mathbb{Z}_{\geq 0}\}$. This problem was suggested to Euler by Goldbach.)