(31.0) **Gamma function.**— The topic of the next two lecture notes is Euler’s Gamma function. Denoted by $\Gamma(z)$, this function was discovered by Euler in 1729, in an attempt to extend the definition of factorial.

The problem of interpolating discrete set of points $\{(n, n!): n \in \mathbb{Z}_{\geq 0}\}$ in $\mathbb{R}^2$ was proposed by Goldback in 1720. More precisely, he asked for a real–valued function of a real variable $f(x)$ such that $f(n) = n!$. Gamma function was defined by Euler as a solution to this problem. I recommend the following survey article for the context and history of Gamma function:


You can download this paper at:

https://people.math.osu.edu/gautam.42/S20/DavisGammaFunction.pdf

(31.1) What is in these notes.—

(1) A definition of $\Gamma(z)$ is given in §31.2 below, using an infinite integral of the kind studied in Lecture 30. We will see a proof of the fact that $\int_0^\infty t^{z-1}e^{-t} \, dt$ defines a holomorphic function on the right half–plane $\{z \in \mathbb{C} : \text{Re}(z) > 0\}$ (denoted by $\mathbb{H}$ below), which we define as $\Gamma(z)$. The proof of the convergence of this infinite integral is optional.

(2) We show that $\Gamma(n) = (n-1)!$ for every $n \in \mathbb{Z}_{\geq 1}$, in §31.3. In §31.4, we prove that $\Gamma(z)$ satisfies the following difference equation: $\Gamma(z+1) = z\Gamma(z)$. This allows us to extend the domain of $\Gamma(z)$, from the right half–plane $\mathbb{H}$ to $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$. Hence, $\Gamma: \mathbb{C} \rightarrow \mathbb{C}$ is a meromorphic function with poles of order 1 at $\mathbb{Z}_{\leq 0}$.

(3) In §31.5, we will use the technique for computing Gaussian integrals to determine the value of $\Gamma(z)$ at $z = \frac{1}{2}$: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(4) As an application of the Gamma function, we compute (real, definite) integrals of the following form: $\int_0^1 x^{p-1}(1-x)^{q-1} \, dx$ in §31.6.

(31.2) **Euler’s integral.**— Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Re}(z) > 0\}$. For $t \in \mathbb{R}_{>0}$, recall that we define: $t^{z-1} = e^{(z-1)\ln(t)}$. Now, consider the following infinite integral:

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1This notation was introduced by Legendre in 1814.
Theorem. This infinite integral defines $\Gamma(z)$ as a holomorphic function on the domain $\mathbb{H}$.

Proof. This proof uses Theorem 30.4 from Lecture 30, page 5. In the notational conventions of Lecture 30, here we have a function $K(t, z) = t^{z-1}e^{-t}$ defined on $(t, z) \in \mathbb{R}_{>0} \times \mathbb{H}$.

In order to prove the theorem, we have to verify Assumptions 1-5 laid out in Lecture 30, §30.4. Note that Assumptions 1-4 hold trivially for our $K(t, z)$. Assumption 5 needs to be checked, both near 0 and $\infty$ since $K(t, z)$ is not defined at $t = 0$. Let us spell out what exactly do we have to prove.

To prove: Given a compact subset $D \subset \mathbb{H}$, and $\varepsilon > 0$, there exist $R > 0$ and $r > 0$ such that:

1. $\left| \int_0^s t^{z-1}e^{-t} \, dt \right| < \varepsilon$ for every $0 < s < r$ and $z \in D$.
2. $\left| \int_S^\infty t^{z-1}e^{-t} \, dt \right| < \varepsilon$ for every $S > R$ and $z \in D$.

Let us choose $A, B \in \mathbb{R}_{>0}$ such that for every $z \in D$, $A < \text{Re}(z) < B$. This can be done, since $D$ is a closed and bounded set contained in $\mathbb{H} = \{ z \in \mathbb{C} : \text{Re}(z) > 0 \}$ (see Figure 1 below).

![Figure 1](image-url)

**Figure 1.** Given a compact set $D$ in the half-plane $\mathbb{H} = \{ z \in \mathbb{C} : \text{Re}(z) > 0 \}$, we can find $A, B \in \mathbb{R}_{>0}$ such that $A < \text{Re}(z) < B$ for every $z \in D$.

Now let us prove (1). Note that since $A > 0$, $x^A = e^{A \text{ln}(x)} \to 0$ as $x \to 0^+$ (i.e, from the right). Choose $r < 1$ small enough so that $r^A < \varepsilon A$ (remember $A$ and $\varepsilon$ are fixed from the

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Optional.
start, so we are just picking some $r < \text{Min}(1, (\varepsilon A) ^ \frac{1}{A})$. For $0 < t < 1$, $\ln(t)$ is negative, which implies:

$$|t^{z-1}| = e^{\Re(z-1)\ln(t)} \leq e^{(A-1)\ln(t)} = t^{A-1} \text{ for } 0 < t < 1, z \in D.$$  

Clearly $e^{-t} < 1$ for $t > 0$. Combining these observations, we have, for every $0 < s < r$:

$$\left| \int_0^s t^{z-1} e^{-t} \, dt \right| \leq \left| \int_0^s t^{A-1} \, dt \right| = \left[ \frac{t^A}{A} \right]_{t=0}^{t=s} = \frac{s^A}{A} < \varepsilon.$$  

This finishes the proof of (1).

Let us prove (2) now. Note that for $t > 1$, we have $\ln(t) > 0$, which implies: $|t^{z-1}| = t^{\Re(z-1)} \leq t^{B-1}$, for $t > 1$ and $z \in D$. Using the fact that $tB^{-1}e^{-\frac{1}{2}} \rightarrow 0$ as $t \rightarrow \infty$, we can choose $t_0 \in \mathbb{R}_{>0}$ be such that

$$t^{B-1}e^{-\frac{1}{2}} \leq 1 \text{ for every } t \geq t_0.$$  

Now pick $R > 1$ to be larger than $t_0$, and such that $e^{-\frac{R}{2}} < \frac{\varepsilon}{2}$. Then, for every $S > R$ we have:

$$\left| \int_S^\infty t^{z-1} e^{-t} \, dt \right| \leq \int_S^\infty (t^{B-1}e^{-\frac{1}{2}}) e^{-\frac{1}{2}} \, dt \leq \int_S^\infty e^{-\frac{1}{2}} \, dt = \left[ -2e^{-\frac{1}{2}} \right]_t^\infty = 2e^{-\frac{S}{2}} < \varepsilon.$$  

\(\square\)

(31.3) Relation with factorial.– As mentioned in Lecture 30, §30.6, this definition generalizes the factorial function $n \mapsto n!$, which is only defined for $n \in \mathbb{Z}_{\geq 0}$ (with the convention that $0! = 1$). To see this, we have the following computation (Exercise 30.6 from Lecture 30).

Claim. For any $n \in \mathbb{Z}_{\geq 0}$, $\int_0^\infty t^n e^{-t} \, dt = n!$. Therefore, we have:

$$\Gamma(n) = (n - 1)! \text{ for every } n \in \mathbb{Z}_{\geq 1}$$
Proof. This proof is by induction on \( n \). For \( n = 0 \), we have:

\[
\int_0^\infty e^{-t} \, dt = \left[ -e^{-t} \right]_{t=0}^{t=\infty} = 1 = 0!
\]

Assuming that the statement has been verified for all \( n = 0, 1, \ldots, \ell \), let us prove it for \( \ell + 1 \).

This step uses integration by parts, and the fact that \( t^N e^{-t} \to 0 \) as \( t \to \infty \), for any \( N \in \mathbb{Z}_{\geq 0} \).

\[
\int_0^\infty t^{\ell+1} e^{-t} \, dt = \left[ -t^{\ell+1} e^{-t} \right]_{t=0}^{t=\infty} + (\ell + 1) \int_0^\infty t^\ell e^{-t} \, dt
\]

\[= 0 + (\ell + 1)! = (\ell + 1)!
\]

\( \Box \)

\[\text{(31.4) A difference equation for } \Gamma(z) \text{.}\]

One of the most important properties of \( \Gamma(z) \) is its behaviour under \( z \mapsto z + 1 \). We have the following equation:

\[
\Gamma(z + 1) = z \Gamma(z)
\]

Proof. This proof is a mild generalization of the one given in the previous section. Namely, we have:

\[
\Gamma(z + 1) - z \Gamma(z) = \int_0^\infty (t^z - z t^{z-1}) e^{-t} \, dt = - \int_0^\infty \frac{d}{dt} \left( t^z e^{-t} \right) \, dt
\]

\[= \left[ -t^z e^{-t} \right]_{t=0}^{t=\infty}
\]

For \( z \) such that \( \text{Re}(z) > 0 \), \( \lim_{t \to \infty} t^z e^{-t} = 0 \), and \( \lim_{t \to 0^+} t^z = 0 \). Hence,

\[
\Gamma(z + 1) - z \Gamma(z) = 0.
\]

\( \Box \)

This relation allows us to extend the domain of \( \Gamma(z) \) to \( \Omega = \mathbb{C} \setminus \{0, -1, -2, \ldots\} = \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \), as follows. A repeated application of \( \Gamma(z + 1) = z \Gamma(z) \), gives us the following, for every \( N \in \mathbb{Z}_{\geq 1} \):

\[
\Gamma(z + N) = (z + N - 1)(z + N - 2) \cdots z \Gamma(z)
\]

Therefore, if we want to define \( \Gamma(z) \) on the half–plane \( \mathbb{H}_{-N} = \{ z \in \mathbb{C} : \text{Re}(z) > -N \} \), then we can set:

\[
\Gamma(z) = \frac{\Gamma(z + N)}{z(z + 1) \cdots (z + N - 1)} = \frac{1}{z(z + 1) \cdots (z + N - 1)} \int_0^\infty t^{z+N-1} e^{-t} \, dt
\]

The last integral being defined, since \( \text{Re}(z + N) > 0 \). Note that, according to this definition, \( \Gamma(z) \) in defined as a meromorphic function, with poles at \( z = 0, -1, -2, \ldots \), which are all of order 1.
Example. Let us compute $\text{Res}_{z=0} (\Gamma(z))$. Since $z = 0$ is pole of order 1, we have:

$$\text{Res}_{z=0} (\Gamma(z)) = \lim_{z \to 0} z \Gamma(z) = \lim_{z \to 0} \Gamma(z+1) = \Gamma(1) = 1.$$ 

where we used that $z \Gamma(z) = \Gamma(z+1)$.

For an arbitrary $n \in \mathbb{Z}_{\geq 1}$, we can compute the residue $\text{Res}_{z=-n} (\Gamma(z))$ by (i) change of variables $w = z + n$ and (ii) repeated application of $\Gamma(z+1) = z \Gamma(z)$:

$$\text{Res}_{z=-n} (\Gamma(z)) = \text{Res}_{w=0} (\Gamma(w-n)) = \text{Res}_{w=0} \left( \frac{\Gamma(w)}{(w-1)(w-2) \cdots (w-n)} \right) = \frac{(-1)^n}{n!}.$$

(31.5) $\Gamma\left(\frac{1}{2}\right)$.— Let us try to compute the value of $\Gamma(z)$ at $z = \frac{1}{2}$. We will see this result again in the next lecture, using another expression for the Gamma function. The answer is:

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Proof. By definition, $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} \, dt$. Consider the change of variables $t = u^2$, which changes this integral to:

$$\Gamma\left(\frac{1}{2}\right) = 2 \int_0^\infty e^{-u^2} \, du = \int_{-\infty}^\infty e^{-u^2} \, du.$$ 

This last integral is known as Gaussian integral and is computed as follows. Let $I = \int_{-\infty}^\infty e^{-u^2} \, du$. Then:

$$I^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-u^2-v^2} \, du \, dv.$$ 

Changing to polar coordinates: $u = r \cos(\theta)$ and $v = r \sin(\theta)$ changes the area element $du \, dv$ to $r \, dr \, d\theta$ (as was done in Calculus III).

$$I^2 = \int_0^{2\pi} \int_0^\infty e^{-r^2} r \, dr \, d\theta = 2\pi \left[ -\frac{e^{-r^2}}{2} \right]_{r=0}^{r=\infty} = \pi$$

Hence, $I = \sqrt{\pi}$ as claimed.

(31.6) Computation of a real integral using Gamma function.— Let $p, q \in \mathbb{R}_{>0}$ and consider the following definite integral

$$B(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} \, dx$$
Remark. If we set \( x = \cos^2(\theta) \), so that \( dx = -2\sin(\theta)\cos(\theta)\,d\theta \), and the limits of the integral become \( \int_{\pi/2}^{0} \), then the integral in question becomes:

\[
B(p, q) = 2 \int_{0}^{\pi/2} \cos^{2p-1}(\theta) \sin^{2q-1}(\theta)\,d\theta
\]

This signifies the use of \( B(p, q) \) in computing various definite integrals involving sines and cosines.

Euler computed the value of \( B(p, q) \) in terms of his Gamma function as:

\[
B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)}
\]

Proof. This proof is a generalization of the computation of the Gaussian integral from the previous section. We begin by writing:

\[
\Gamma(p)\Gamma(q) = \int_{0}^{\infty} \int_{0}^{\infty} t_1^{p-1}e^{-t_1}t_2^{q-1}e^{-t_2}\,dt_1\,dt_2
\]

Change of variables: \( t_1 = u_1^2 \) and \( t_2 = u_2^2 \) allows us to write it as:

\[
\Gamma(p)\Gamma(q) = 4 \int_{0}^{\infty} \int_{0}^{\infty} e^{-u_1^2-u_2^2}u_1^{2p-1}u_2^{2q-1}\,du_1\,du_2
\]

Set \( u_1 = r\cos(\theta) \) and \( u_2 = r\sin(\theta) \) as in the previous section. Since \((u_1, u_2)\) are in the first quadrant of \( \mathbb{R}^2 \), the limits of integration are: \( 0 < r < \infty \) and \( 0 \leq \theta \leq \frac{\pi}{2} \). We get:

\[
\Gamma(p)\Gamma(q) = \frac{\Gamma(p + q)}{2} \left( \int_{0}^{\infty} e^{-r^2}r^{2(p+q)-1}\,dr \right) \cdot \left( \int_{0}^{\pi/2} \cos^{2p-1}(\theta) \sin^{2q-1}(\theta)\,d\theta \right)
\]

The first term gives us \( \frac{\Gamma(p + q)}{2} \), since upon setting \( r^2 = t \) we have:

\[
\int_{0}^{\infty} e^{-r^2}r^{2(p+q)-1}\,dr = \frac{1}{2} \int_{0}^{\infty} e^{-t}t^{p+q-1}\,dt = \frac{\Gamma(p + q)}{2}.
\]

And the second term gives us \( \frac{B(p, q)}{2} \) (see the remark above). Hence:

\[
\Gamma(p)\Gamma(q) = \Gamma(p + q)B(p, q)
\]
as claimed. \( \square \)