COMPLEX ANALYSIS: LECTURE 31

(31.0) Gamma function.– The topic of the next two lecture notes is Euler's Gamma function. Denoted by $\Gamma(z)$[1], this function was discovered by Euler in 1729, in an attempt to extend the definition of factorial.

The problem of interpolating discrete set of points $\{(n, n!) : n \in \mathbb{Z}_{\geq 0}\}$ in $\mathbb{R}^2$ was proposed by Goldback in 1720. More precisely, he asked for a real–valued function of a real variable $f(x)$ such that $f(n) = n!$ Gamma function was defined by Euler as a solution to this problem. I recommend the following survey article for the context and history of Gamma function:

Philip J. Davis. *Leonhard Euler's integral: a historical profile of gamma function.* The American Mathematical Monthly vol. 66 (1959), 849–869.

You can download this paper at:

`https://people.math.osu.edu/gautam.42/S20/DavisGammaFunction.pdf`

(31.1) What is in these notes.–

(1) A definition of $\Gamma(z)$ is given in §31.2 below, using an infinite integral of the kind studied in Lecture 30. We will see a proof of the fact that $\displaystyle\int_0^\infty t^{z-1}e^{-t}\,dt$ defines a holomorphic function on the right half–plane $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ (denoted by $\mathbb{H}$ below), which we define as $\Gamma(z)$. The proof of the convergence of this infinite integral is optional.

(2) We show that $\Gamma(n) = (n-1)!$ for every $n \in \mathbb{Z}_{\geq 1}$, in §31.3. In §31.4, we prove that $\Gamma(z)$ satisfies the following difference equation: $\boxed{\Gamma(z+1) = z\Gamma(z)}$. This allows us to extend the domain of $\Gamma(z)$, from the right half–plane $\mathbb{H}$ to $\mathbb{C} \setminus \{0, -1, -2, \ldots\}$. Hence, $\Gamma : \mathbb{C} \dashrightarrow \mathbb{C}$ is a *meromorphic function* with poles of order 1 at $\mathbb{Z}_{\leq 0}$.

(3) In §31.5, we will use the technique for computing Gaussian integrals to determine the value of $\Gamma(z)$ at $z = \frac{1}{2}$: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(4) As an application of the Gamma function, we compute (real, definite) integrals of the following form: $\displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}\,dx$ in §31.6.

(31.2) Euler's integral.– Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. For $t \in \mathbb{R}_{>0}$, recall that we define: $t^{z-1} = e^{(z-1)\ln(t)}$. Now, consider the following infinite integral:

[1]This notation was introduced by Legendre in 1814.

$$\boxed{\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt}$$

Theorem. *This infinite integral defines $\Gamma(z)$ as a holomorphic function on the domain $\mathbb{H}$.*

Proof. [2] This proof uses Theorem 30.4 from Lecture 30, page 5. In the notational conventions of Lecture 30, here we have a function $K(t,z) = t^{z-1}e^{-t}$ defined on $(t,z) \in \mathbb{R}_{>0} \times \mathbb{H}$.

In order to prove the theorem, we have to verify Assumptions 1-5 laid out in Lecture 30, §30.4. Note that Assumptions 1-4 hold trivially for our $K(t,z)$. Assumption 5 needs to be checked, both near 0 and ∞ since $K(t,z)$ is not defined at $t = 0$. Let us spell out what exactly do we have to prove.

To prove: Given a compact subset $D \subset \mathbb{H}$, and $\varepsilon > 0$, there exist $R > 0$ and $r > 0$ such that:

$$(1) \quad \left| \int_0^s t^{z-1} e^{-t}\, dt \right| < \varepsilon \text{ for every } 0 < s < r \text{ and } z \in D.$$

$$(2) \quad \left| \int_S^\infty t^{z-1} e^{-t}\, dt \right| < \varepsilon \text{ for every } S > R \text{ and } z \in D.$$

Let us choose $A, B \in \mathbb{R}_{>0}$ such that for every $z \in D$, $A < \mathrm{Re}(z) < B$. This can be done, since D is a closed and bounded set contained in $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ (see Figure 1 below).

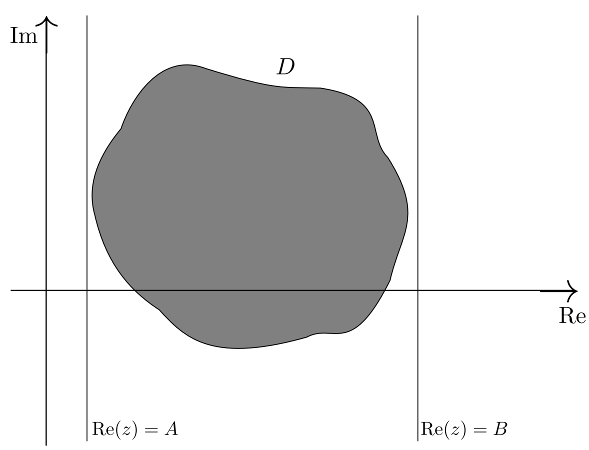

FIGURE 1. Given a compact set D in the half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$, we can find $A, B \in \mathbb{R}_{>0}$ such that $A < \mathrm{Re}(z) < B$ for every $z \in D$.

Now let us prove (1). Note that since $A > 0$, $x^A = e^{A\ln(x)} \to 0$ as $x \to 0^+$ (i.e, from the right). Choose $r < 1$ small enough so that $r^A < \varepsilon A$ (remember A and ε are fixed from the

[2] Optional.

start, so we are just picking some $r < \text{Min}(1, (\varepsilon A)^{\frac{1}{A}}))$. For $0 < t < 1$, $\ln(t)$ is negative, which implies:

$$|t^{z-1}| = e^{\text{Re}\,(z-1)\ln(t)} \leq e^{(A-1)\ln(t)} = t^{A-1} \text{ for } 0 < t < 1, z \in D.$$

Clearly $e^{-t} < 1$ for $t > 0$. Combining these observations, we have, for every $0 < s < r$:

$$\left|\int_0^s t^{z-1}e^{-t}\,dt\right| \leq \left|\int_0^s t^{A-1}\,dt\right| = \left[\frac{t^A}{A}\right]_{t=0}^{t=s} = \frac{s^A}{A} < \varepsilon.$$

This finishes the proof of (1).

Let us prove (2) now. Note that for $t > 1$, we have $\ln(t) > 0$, which implies: $|t^{z-1}| = t^{\text{Re}(z-1)} \leq t^{B-1}$, for $t > 1$ and $z \in D$. Using the fact that $t^{B-1}e^{-\frac{t}{2}} \to 0$ as $t \to \infty$, we can choose $t_0 \in \mathbb{R}_{>0}$ be such that

$$t^{B-1}e^{-\frac{t}{2}} \leq 1 \text{ for every } t \geq t_0.$$

Now pick $R > 1$ to be larger than t_0, and such that $e^{-\frac{R}{2}} < \frac{\varepsilon}{2}$. Then, for every $S > R$ we have:

$$\left|\int_S^\infty t^{z-1}e^{-t}\,dt\right| \leq \int_S^\infty \left(t^{B-1}e^{-\frac{t}{2}}\right)e^{-\frac{t}{2}}\,dt \leq \int_S^\infty e^{-\frac{t}{2}}\,dt = \left[-2e^{-\frac{t}{2}}\right]_{t=S}^{\infty}$$
$$= 2e^{-\frac{S}{2}} < \varepsilon.$$

□

(31.3) Relation with factorial.– As mentioned in Lecture 30, §30.6, this definition generalizes the factorial function $n \mapsto n!$, which is only defined for $n \in \mathbb{Z}_{\geq 0}$ (with the convention that $0! = 1$). To see this, we have the following computation (Exercise 30.6 from Lecture 30).

Claim. For any $n \in \mathbb{Z}_{\geq 0}$, $\displaystyle\int_0^\infty t^n e^{-t}\,dt = n!$. Therefore, we have:

$$\boxed{\Gamma(n) = (n-1)! \text{ for every } n \in \mathbb{Z}_{\geq 1}}$$

Proof. This proof is by induction on n. For $n = 0$, we have:

$$\int_0^\infty e^{-t}\,dt = \left[-e^{-t}\right]_{t=0}^{t=\infty} = 1 = 0!$$

Assuming that the statement has been verified for all $n = 0, 1, \dots \ell$, let us prove it for $\ell + 1$. This step uses integration by parts, and the fact that $t^N e^{-t} \to 0$ as $t \to \infty$, for any $N \in \mathbb{Z}_{\geq 0}$.

$$\begin{aligned}\int_0^\infty t^{\ell+1} e^{-t}\,dt &= \left[-t^{\ell+1}e^{-t}\right]_{t=0}^{t=\infty} + (\ell+1)\int_0^\infty t^\ell e^{-t}\,dt \\ &= 0 + (\ell+1)\ell! = (\ell+1)!\end{aligned}$$

□

(31.4) A differerence equation for $\Gamma(z)$.– One of the most important properties of $\Gamma(z)$ is its behaviour under $z \mapsto z+1$. We have the following equation:

$$\boxed{\Gamma(z+1) = z\Gamma(z)}$$

Proof. This proof is a mild generalization of the one given in the previous section. Namely, we have:

$$\begin{aligned}\Gamma(z+1) - z\Gamma(z) &= \int_0^\infty (t^z - zt^{z-1})e^{-t}\,dt = -\int_0^\infty \frac{d}{dt}\left(t^z e^{-t}\right)\,dt \\ &= \left[-t^z e^{-t}\right]_{t=0}^{t=\infty}\end{aligned}$$

For z such that $\mathrm{Re}(z) > 0$, $\lim\limits_{t\to\infty} t^z e^{-t} = 0$, and $\lim\limits_{t\to 0^+} t^z = 0$. Hence,

$$\Gamma(z+1) - z\Gamma(z) = 0.$$

□

This relation allows us to *extend* the domain of $\Gamma(z)$ to $\Omega = \mathbb{C} \setminus \{0, -1, -2, \dots\} = \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, as follows. A repeated application of $\Gamma(z+1) = z\Gamma(z)$, gives us the following, for every $N \in \mathbb{Z}_{\geq 1}$:

$$\Gamma(z+N) = (z+N-1)(z+N-2)\cdots z\Gamma(z)$$

Therefore, if we want to define $\Gamma(z)$ on the half–plane $\mathbb{H}_{-N} = \{z \in \mathbb{C} : \mathrm{Re}(z) > -N\}$, then we can set:

$$\Gamma(z) = \frac{\Gamma(z+N)}{z(z+1)\cdots(z+N-1)} = \frac{1}{z(z+1)\cdots(z+N-1)}\int_0^\infty t^{z+N-1}e^{-t}\,dt$$

The last integral being defined, since $\mathrm{Re}(z+N) > 0$. Note that, according to this definition, $\Gamma(z)$ in defined as a *meromorphic* function, with poles at $z = 0, -1, -2, \dots$, which are all of order 1.

Example. Let us compute $\operatorname*{Res}_{z=0}(\Gamma(z))$. Since $z=0$ is pole of order 1, we have:

$$\operatorname*{Res}_{z=0}(\Gamma(z)) = \lim_{z\to 0} z\Gamma(z) = \lim_{z\to 0}\Gamma(z+1) = \Gamma(1) = 1.$$

where we used that $z\Gamma(z) = \Gamma(z+1)$.

For an arbitrary $n \in \mathbb{Z}_{\geq 1}$, we can compute the residue $\operatorname*{Res}_{z=-n}(\Gamma(z))$ by (i) change of variables $w = z+n$ and (ii) repeated application of $\Gamma(z+1) = z\Gamma(z)$:

$$\operatorname*{Res}_{z=-n}(\Gamma(z)) = \operatorname*{Res}_{w=0}(\Gamma(w-n)) = \operatorname*{Res}_{w=0}\left(\frac{\Gamma(w)}{(w-1)(w-2)\cdots(w-n)}\right) = \frac{(-1)^n}{n!}.$$

(31.5) $\Gamma\left(\frac{1}{2}\right)$**.–** Let us try to compute the value of $\Gamma(z)$ at $z = \frac{1}{2}$. We will see this result again in the next lecture, using another expression for the Gamma function. The answer is:

$$\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$$

Proof. By definition, $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty t^{-\frac{1}{2}} e^{-t}\, dt$. Consider the change of variables $t = u^2$, which changes this integral to:

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\, du = \int_{-\infty}^\infty e^{-u^2}\, du.$$

This last integral is known as *Gaussian integral* and is computed as follows. Let $I = \int_{-\infty}^\infty e^{-u^2}\, du$. Then:

$$I^2 = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-u^2-v^2}\, du\, dv.$$

Changing to polar coordinates: $u = r\cos(\theta)$ and $v = r\sin(\theta)$ changes the area element $du\ dv$ to $r\ dr\ d\theta$ (as was done in Calculus III).

$$I^2 = \int_0^{2\pi}\int_0^\infty e^{-r^2} r\ dr\ d\theta = 2\pi\left[-\frac{e^{-r^2}}{2}\right]_{r=0}^{r=\infty} = \pi$$

Hence, $I = \sqrt{\pi}$ as claimed. □

(31.6) Computation of a real integral using Gamma function.– Let $p, q \in \mathbb{R}_{>0}$ and consider the following definite integral

$$\boxed{B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\, dx}$$

Remark. If we set $x = \cos^2(\theta)$, so that $dx = -2\sin(\theta)\cos(\theta)\,d\theta$, and the limits of the integral become $\int_{\frac{\pi}{2}}^{0}$, then the integral in question becomes:

$$B(p,q) = 2\int_0^{\frac{\pi}{2}} \cos^{2p-1}(\theta)\sin^{2q-1}(\theta)\,d\theta$$

This signifies the use of $B(p,q)$ in computing various definite integrals involving sines and cosines.

Euler computed the value of $B(p,q)$ in terms of his Gamma function as:

$$\boxed{B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}}$$

Proof. This proof is a generalization of the computation of the Gaussian integral from the previous section. We begin by writing:

$$\Gamma(p)\Gamma(q) = \int_0^\infty \int_0^\infty t_1^{p-1}e^{-t_1}t_2^{q-1}e^{-t_2}\,dt_1\,dt_2$$

Change of variables: $t_1 = u_1^2$ and $t_2 = u_2^2$ allows us to write it as:

$$\Gamma(p)\Gamma(q) = 4\int_0^\infty \int_0^\infty e^{-u_1^2-u_2^2}u_1^{2p-1}u_2^{2q-1}\,du_1\,du_2$$

Set $u_1 = r\cos(\theta)$ and $u_2 = r\sin(\theta)$ as in the previous section. Since (u_1, u_2) are in the first quadrant of $\mathbb{R}^2$, the limits of integration are: $0 < r < \infty$ and $0 \le \theta \le \frac{\pi}{2}$. We get:

$$\Gamma(p)\Gamma(q) = 4\left(\int_0^\infty e^{-r^2}r^{2(p+q)-1}\,dr\right)\cdot\left(\int_0^{\frac{\pi}{2}} \cos^{2p-1}(\theta)\sin^{2q-1}(\theta)\,d\theta\right)$$

The first term gives us $\frac{\Gamma(p+q)}{2}$, since upon setting $r^2 = t$ we have:

$$\int_0^\infty e^{-r^2}r^{2(p+q)-1}\,dr = \frac{1}{2}\int_0^\infty e^{-t}t^{p+q-1}\,dt = \frac{\Gamma(p+q)}{2}.$$

And the second term gives us $\frac{B(p,q)}{2}$ (see the remark above). Hence:

$$\Gamma(p)\Gamma(q) = \Gamma(p+q)B(p,q)$$

as claimed. □