COMPLEX ANALYSIS: LECTURE 32

(32.0) Gamma function continued.– Recall that in Lecture 31 we defined $\Gamma(z)$ via the following infinite integral:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt.$$

This infinite integral converges uniformly on the right half–plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ and thus defines a holomorphic function. (Theorem 31.2).

Now we will study a different expression of the Gamma function, obtained by Weierstrass in 1856. To keep the notation separate for now (until it is proved that what is written below is the Gamma function - in §32.4), we will denote the function defined by Weierstrass as $\Gamma_1(z)$:

$$\boxed{\frac{1}{\Gamma_1(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \right\}}$$

where $\gamma = 0.5772157\ldots$ is a constant (called Euler–Mascheroni constant, defined in §32.1 below).

You should compare this expression with the ones that appeared in Lecture 29, §29.3. In §32.2 below, we will prove that this infinite product converges uniformly (over the entire complex plane), and hence defines a holomorphic function $\mathbb{C} \to \mathbb{C}$.

Weierstrass' expression makes it easy to connect the Gamma function with trigonometric functions. The following identity is proved in §32.4, using the infinite product expansion of $\sin(z)$ obtained in Lecture 29, §29.4.

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}}$$

(32.1) Euler–Mascheroni constant.– The number γ is defined as follows:

$$\gamma = \lim_{N\to\infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} - \ln(N+1)\right)$$

To see that the limit written above exists, we can proceed as follows. Consider the integral $u_n = \displaystyle\int_0^1 \frac{t}{n(t+n)}\, dt$. Its value can be computed by writing the partial fraction

$$\int_0^1 \frac{t}{n(t+n)}\, dt = \int_0^1 \left(\frac{1}{n} - \frac{1}{t+n}\right) dt = \frac{1}{n} - \ln(n+1) + \ln(n)$$

Thus, the sum $\sum_{n=1}^{N} u_n = 1 + \frac{1}{2} + \cdots + \frac{1}{N} - \ln(N+1)$. On the other hand, for $0 < t < 1$, we have $\left|\frac{t}{n(t+n)}\right| \leq \frac{1}{n^2}$. This implies that $u_n \leq \frac{1}{n^2}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (its value is $\frac{\pi^2}{6}$ from Lecture 29, §29.0), by comparison test we conclude that $\sum_{n=1}^{\infty} u_n$ is convergent as well.

(32.2) Weierstrass' infinite product expression.– Consider the following product

$$G(z) = \prod_{n=1}^{\infty} \left\{ \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}} \right\}$$

Theorem. *This infinite product converges uniformly and thus defines a holomorphic function* $G : \mathbb{C} \to \mathbb{C}$. *It has zeroes of order* 1 *at* $z = -1, -2, \ldots$

Proof. [1] Let $D \subset \mathbb{C}$ be a compact subset. Choose N_0 large enough so that $\frac{|z|}{N_0} < \frac{1}{2}$. Then, for every $n \geq N_0$, we can estimate $\left|\ln\left(1 + \frac{z}{n}\right) - \frac{z}{n}\right|$ as follows. Here, recall that $\ln(w) = \ln(|w|) + \mathbf{i}\arg(w)$, is defined on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, with $-\pi < \arg(w) < \pi$. Since $\frac{|z|}{n} < \frac{1}{2}$, $\ln\left(1 + \frac{z}{n}\right)$ is defined, and is given by the following Taylor series:

$$\ln\left(1 + \frac{z}{n}\right) = \frac{z}{n} - \frac{1}{2}\frac{z^2}{n^2} + \frac{1}{3}\frac{z^3}{n^3} - \cdots$$

This implies:

$$\begin{aligned} \left|\ln\left(1 + \frac{z}{n}\right) - \frac{z}{n}\right| &= \left|-\frac{1}{2}\frac{z^2}{n^2} + \frac{1}{3}\frac{z^3}{n^3} - \cdots\right| \leq \frac{1}{2}\frac{|z|^2}{n^2}\left(1 + \frac{|z|}{n} + \frac{|z|^2}{n^2} + \cdots\right) \\ &< \frac{1}{2}\frac{4N_0^2}{n^2}\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) = \frac{N_0^2}{n^2}. \end{aligned}$$

Therefore, $\left|\sum_{n=N_0+1}^{\infty}\left(\ln\left(1 + \frac{z}{n}\right) - \frac{z}{n}\right)\right| < N_0^2 \cdot \left(\sum_{n=N_0+1}^{\infty} \frac{1}{n^2}\right)$ converges for every $z \in D$. Hence it defines a holomorphic function. This implies the theorem, since we can write

$$G(z) = \left(\prod_{n=1}^{N_0} \left\{\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}\right\}\right) \cdot \exp\left(\sum_{n=N_0+1}^{\infty}\left(\ln\left(1 + \frac{z}{n}\right) - \frac{z}{n}\right)\right).$$

□

[1] Optional. This proof was promised in Lecture 29, §29.3, page 7.

(32.3) Weierstrass' formula for the Gamma function..– Let $G:\mathbb{C}\to\mathbb{C}$ be the holomorphic function from Theorem 32.2 above, with zeroes of order 1 at $z\in\mathbb{Z}_{\leq -1}$. Define a function $\Gamma_1(z)$ by the following equation:

$$\boxed{\frac{1}{\Gamma_1(z)} = ze^{\gamma z}\cdot G(z)}$$

where γ is the constant from §32.1 above. Thus, $\Gamma_1(z)$ is a meromorphic function with poles of order 1 at $z\in\mathbb{Z}_{\leq 0}$.

Example. Let us compute $G(1)$ from the definition: $G(z)=\displaystyle\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\}$. Thus $G(z)$ is the uniform limit of $\{G_N(z)\}_{N=1}^{\infty}$, where

$$G_N(z)=\prod_{n=1}^{N}\left\{\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}}\right\} = \frac{(z+1)(z+2)\cdots(z+N)}{1\cdot 2\cdots N}e^{-\left(1+\frac{1}{2}+\cdots\frac{1}{N}\right)z}.$$

Setting $z=1$, we get:

$$G_N(1)=(N+1)e^{-\sum_{n=1}^{N}\frac{1}{n}} = e^{-\left(\sum_{n=1}^{N}\frac{1}{n}-\ln(N+1)\right)}$$

By definition of γ (see §32.1 above), we obtain $G(1)=\lim\limits_{N\to\infty}G_N(1)=e^{-\gamma}$. Hence, $\Gamma_1(1)=1$.

(32.4) $\Gamma_1=\Gamma$.– Our very first task here is to prove that $\Gamma_1(z)$ is given by the Eulerian integral $\displaystyle\int_0^{\infty}t^{z-1}e^{-t}\,dt$, when $\mathrm{Re}(z)>0$. Thus on the right half–plane $\mathbb{H}=\{z\in\mathbb{C}:\mathrm{Re}(z)>0\}$, we have the equality of (holomorphic) functions:

$$\Gamma_1(z)=\Gamma(z)=\int_0^{\infty}t^{z-1}e^{-t}\,dt \text{ for } z\in\mathbb{H}.$$

This proof proceeds in two steps[2]. We will establish that both sides of the equation are equal to the following limit (we are going to keep $z\in\mathbb{H}$ fixed throughout):

$$\lim_{N\to\infty}\frac{N!}{z(z+1)\cdots(z+N)}N^z\ .$$

Step 1. $\Gamma_1(z)=\lim\limits_{N\to\infty}\dfrac{N!}{z(z+1)\cdots(z+N)}N^z$.

Recall that we just defined $\Gamma_1(z)=\lim\limits_{N\to\infty}\dfrac{e^{-\gamma z}}{z}\dfrac{1}{G_N(z)}$, where (see Example in §32.3 above):

$$G_N(z)=\frac{(z+1)(z+2)\cdots(z+N)}{N!}e^{-z\sum_{n=1}^{N}\frac{1}{n}}.$$

[2]Optional.

Hence $\Gamma_1(z) = \lim_{N\to\infty} \dfrac{N!}{z(z+1)\cdots(z+N)} \exp\left(z\left(\sum_{n=1}^{N}\dfrac{1}{n}\right) - z\gamma\right)$.

Replacing $\left(\sum_{n=1}^{N}\dfrac{1}{n}\right) - \gamma$ by $\ln(N)$ (using the definition of γ from §32.1), we get:

$$\Gamma_1(z) = \lim_{N\to\infty} \frac{N!}{z(z+1)\cdots(z+N)} e^{z\ln(N)} = \lim_{N\to\infty} \frac{N!}{z(z+1)\cdots(z+N)} N^z ,$$

as claimed.

Step 2. $\Gamma(z) = \lim_{N\to\infty} \dfrac{N!}{z(z+1)\cdots(z+N)} N^z$.

This computation also goes back to Euler. The idea is to replace e^{-t} by $\left(1-\frac{t}{N}\right)^N$ and integrate by parts.

Consider the integral $P(z,N) = \displaystyle\int_0^N t^{z-1}\left(1-\frac{t}{N}\right)^N dt$. A simple change of variables $t = N\tau$ gives us:

$$P(z,N) = N^z \int_0^1 (1-\tau)^N \tau^{z-1}\, d\tau$$

Integration by parts implies:

$$\int_0^1 (1-\tau)^N \tau^{z-1}\, d\tau = \left[(1-\tau)^N \frac{\tau^z}{z}\right]_{\tau=0}^{\tau=1} + \frac{N}{z}\int_0^1 (1-\tau)^{N-1}\tau^z\, d\tau$$
$$= \frac{N}{z}\int_0^1 (1-\tau)^{N-1}\tau^z\, d\tau.$$

Repeating this calculation N times, we conclude:

$$\int_0^1 (1-\tau)^N \tau^{z-1}\, d\tau = \frac{N(N-1)\cdots 1}{z(z+1)\cdots(z+N-1)} \int_0^1 \tau^{z+N-1}\, d\tau = \frac{N!}{z(z+1)\cdots(z+N)}$$

Hence, $P(z,N) = \dfrac{N!}{z(z+1)\cdots(z+N)} N^z$. It remains to show that $\Gamma(z) = \lim_{N\to\infty} P(z,N)$. Since $\Gamma(z) = \lim_{N\to\infty}\int_0^N t^{z-1}e^{-t}\, dt$, the difference of the two terms can be written as:

$$\Gamma(z) - \lim_{N\to\infty} P(z,N) = \lim_{N\to\infty}\int_0^N t^{z-1}\left(e^{-t} - \left(1-\frac{t}{N}\right)^N\right) dt$$

We are going to need the following inequality:

Inequality. For every $N \in \mathbb{Z}_{\geq 1}$ and $0 \leq t \leq N$, we have

$$\boxed{0 \leq e^{-t} - \left(1-\frac{t}{N}\right)^N \leq \frac{1}{N}t^2 e^{-t}}$$

Assuming this, we can finish the proof as follows. Let $x = \mathrm{Re}(z) > 0$:

$$\left|\int_0^N t^{z-1}\left(e^{-t} - \left(1 - \frac{t}{N}\right)^N\right) dt\right| \leq \int_0^N \frac{1}{N} t^{x+1} e^{-t}\, dt < \frac{1}{N}\int_0^\infty t^{x+1}e^{-t}\, dt$$
$$= \frac{\Gamma(x+1)}{N} \to 0 \text{ as } N \to \infty.$$

Hence, $\Gamma(z) = \lim_{N\to\infty} P(z, N)$. This finishes the proof, modulo the inequality stated above. Its proof is given in §32.6 below.

(32.5) Relation with trigonometric functions.– In Lecture 29, §29.4, the following identity is proved:

$$\frac{\sin(z)}{z} = \prod_{n\in\mathbb{Z}_{\neq 0}} \left\{\left(1 - \frac{z}{n\pi}\right) e^{\frac{z}{n\pi}}\right\}$$

Replace z by πz to rewrite this as:

$$\frac{\sin(\pi z)}{\pi z} = \left(\prod_{n=1}^\infty \left\{\left(1 - \frac{z}{n}\right) e^{\frac{z}{n}}\right\}\right) \cdot \left(\prod_{n=1}^\infty \left\{\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}\right\}\right)$$
$$= G(-z)G(z), \quad \text{where } G(z) \text{ was defined in §32.2 above.}$$

Hence, we obtain:

$$\frac{\sin(\pi z)}{\pi z} = \frac{1}{-ze^{-\gamma z}\Gamma(-z)} \cdot \frac{1}{ze^{\gamma z}\Gamma(z)}, \quad \text{since } ze^{\gamma z}G(z) = \frac{1}{\Gamma(z)}.$$

Taking inverse on both sides, and using the fact that $-z\Gamma(-z) = \Gamma(1-z)$, we obtain the following relation between the Gamma function and trigonometric functions:

$$\boxed{\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}}$$

Example. Set $z = \frac{1}{2}$ in the equation above. This gives us a different proof of $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(32.6) Proof of the inequality from §32.4.– Recall the inequality claimed in §32.4. For every $N \in \mathbb{Z}_{\geq 1}$ and $0 \leq t \leq N$, we have:

$$0 \leq e^{-t} - \left(1 - \frac{t}{N}\right)^N \leq \frac{1}{N}t^2 e^{-t}$$

In order to obtain this, let us consider $0 \leq y \leq 1$. By comparing the Taylor series expansions, it is easy to see that

$$1 + y \leq e^y \leq \frac{1}{1-y}.$$

Now set $y = \frac{t}{N}$ to get:

$$\left(1 + \frac{t}{N}\right)^N \leq e^t \leq \left(1 - \frac{t}{N}\right)^{-N}$$

and taking inverses of each term (which flips the inequality):

$$\left(1-\frac{t}{N}\right)^N \le e^{-t} \le \left(1+\frac{t}{N}\right)^{-N}$$

Thus we obtain $0 \le e^{-t} - \left(1-\frac{t}{N}\right)^N$. Moreover,

$$\begin{aligned}
e^{-t} - \left(1-\frac{t}{N}\right)^N &\le e^{-t}\left(1 - e^t\left(1-\frac{t}{N}\right)^N\right) \\
&\le e^{-t}\left(1 - \left(1+\frac{t}{N}\right)^N\left(1-\frac{t}{N}\right)^N\right) \quad \left(\text{since } e^t \ge \left(1+\frac{t}{N}\right)^N\right) \\
&= e^{-t}\left(1-\left(1-\frac{t^2}{N^2}\right)^N\right) \le e^{-t}\frac{t^2}{N}.
\end{aligned}$$

In the last step, I have used the fact that for $0 \le y \le 1$, we have $(1-y)^n \ge 1-ny$. This is clear when $ny \ge 1$, and can be shown by induction on n, in case $ny < 1$:

$$\begin{aligned}
(1-y)^{n+1} &= (1-y)^n(1-y) \ge (1-ny)(1-y) \\
&= 1-(n+1)y + ny^2 \ge 1-(n+1)y.
\end{aligned}$$