COMPLEX ANALYSIS: LECTURE 33

(33.0) Doubly–periodic functions.– This is the last topic of our course. We are going to study functions which are periodic with respect to two complex numbers.

If $f(z)$ is a holomorphic function which satisfies $f(z+c) = f(z)$, for some complex number $c \in \mathbb{C}$, we say that f is *periodic with respect to* c, or c is a *period of* f. For instance, $e^z = e^{z+2\pi\mathbf{i}}$. Therefore, e^z is periodic with period $2\pi\mathbf{i}$. Similarly, $\sin(z), \cos(z)$ are periodic with period 2π.

From now on, we are going to fix a complex number τ in the upper half of the complex plane. That is, $\mathrm{Im}(\tau) > 0$. Define:

$$\Lambda_\tau = \{m + n\tau : m, n \in \mathbb{Z}\} \subset \mathbb{C} \ .$$

Definition. A doubly–periodic function, with respect to Λ_τ, is a meromorphic function $f : \mathbb{C} \dashrightarrow \mathbb{C}$ such that

$$\boxed{f(z+\ell) = f(z)} \text{ for every } \ell \in \Lambda_\tau \ .$$

In other words, $f(z+1) = f(z)$ and $f(z+\tau) = f(z)$. It has two independent [1] periods 1 and τ, hence the name *doubly–periodic*.

Remark. Doubly–periodic functions are also known as *elliptic functions*, because of their significance in computing arc length of an ellipse.

Summary of results in these notes. In this set of notes, we will prove some basic properties every doubly–periodic function must have. These are necessary conditions which tell us what cannot be expected from doubly–periodic functions. For instance,

(1) We cannot have a non–constant, holomorphic doubly–periodic function (Theorem 33.2).

(2) There is no doubly–periodic function with only one pole of order 1 within a fundamental parallelogram (see §33.1 for the definition of a fundamental parallelogram, and Theorem 33.4 (1)).

Theorem 33.4 contains three properties of a doubly–periodic function, contraining the set of zeroes and poles it can have. These conditions also turn out to be sufficient, classifying

[1] Independent over $\mathbb{R}$, that is $\{1, \tau\}$ are linearly independent elements of $\mathbb{C} = \mathbb{R}^2$ over $\mathbb{R}$. This is because $\mathrm{Im}(\tau) \neq 0$.

the whole family of doubly–periodic functions, as we will see in Lecture 34.

Caveat: These notes do not contain any example of a doubly–periodic function. We will construct such examples with the help of a special function known as *Jacobi's theta function* in the next lecture.

(33.1) Fundamental parallelogram.–

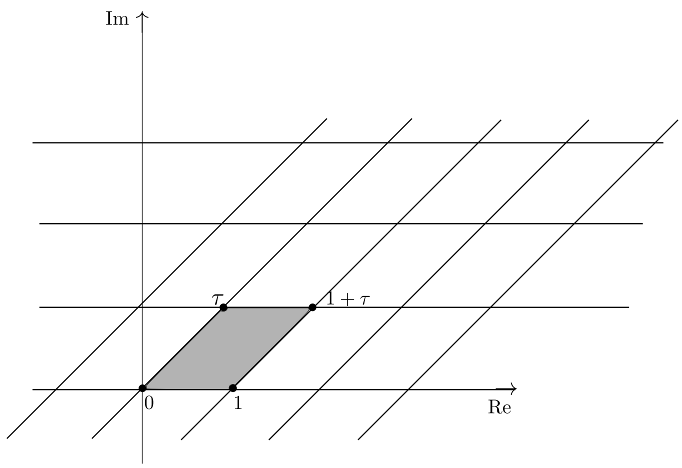

FIGURE 1. Grid generated by 1 and τ. Λ_τ is the set of vertices $\{m + n\tau : m, n \in \mathbb{Z}\}$. A fundamental parallelogram is shaded.

Given any complex number $t \in \mathbb{C}$, we can draw a parallelogram with vertices $\{t, t+1, t+\tau, t+1+\tau\}$. Any such parallelogram is called a *fundamental parallelogram*. See Figure 1 above.

Given two complex numbers $z_1, z_2 \in \mathbb{C}$, we say:$z_1 \equiv z_2$ (modulo Λ_τ) if $z_1 - z_2 \in \Lambda_\tau$. So, if $f(z)$ is a doubly–periodic function with respect to Λ_τ, and $z_1 \equiv z_2$ (modulo Λ_τ), then $f(z_1) = f(z_2)$.

Note that, if R is a fundamental parallelogram, then given any $z \in \mathbb{C}$, we can find z^* in R such that $z \equiv z^*$ (modulo Λ_τ). Hence, the behaviour of a doubly–periodic function $f(z)$ with respect to Λ_τ is completely determined by its behaviour on a fundamental parallelogram.

(33.2) Holomorphic doubly–periodic functions are constants.– Let $f(z)$ be a doubly–periodic function with respect to Λ_τ. Assume that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic (so f has no singularities anywhere on the complex plane).

Let R be a fundamental parallelogram (for instance, with vertices $0, 1, \tau$ and $1+\tau$, as shown in Figure 1 above). Since R is closed and bounded, there exists a contant $M \in \mathbb{R}_{>0}$ such that $|f(z)| \leq M$ for every $z \in R$.

As $f(z)$ is doubly–periodic, we get that $|f(z)| \leq M$ for every $z \in \mathbb{C}$. Hence, $f(z)$ is an entire holomorphic function, which is bounded. By Liouville's theorem (see Lecture 18,

§18.2, page 3) any bounded entire function has to be constant. Hence, we have proved:

Theorem. *Any doubly–periodic holomorphic function is constant.*

(33.3) Zeroes and poles within a fundamental parallelogram.– Now we know that any non–constant doubly–periodic function $f(z)$ must have a singularity. By our definition, $f(z)$ is assumed to be meromorphic, meaning all its singularities are poles (recall from Lecture 26, §26.5, page 4 - meromorphic functions are not allowed to have essential singularities).

Applying the same logic to $\dfrac{1}{f(z)}$, we conclude that $f(z)$ must have a zero.

Theorem. *Let R be a fundamental parallelogram. Then $f(z)$ has finitely many zeroes and finitely many poles in R.*

Proof. Since R is a closed and bounded set and $f(z)$ is meromorphic, the set of its poles within R has to be finite. This argument was given in Lecture 26, Proposition 26.5, page 5. Let us quickly review it here.

If $f(z)$ had infinitely many poles in R, then these poles would accumulate near a point in R, which then would have to be an essential singularity (see Lecture 26, §26.3, page 3). It would then be a contradiction to the hypothesis that $f(z)$ is meromorphic.

Hence, we know that $f(z)$ can only have finitely many poles within a fundamental parallelogram. Similarly, carrying out this argument for $\dfrac{1}{f(z)}$, we conclude that $f(z)$ can only have finitely many zeroes within a fundamental parallelogram. □

(33.4) Constraints on residues and number of zeroes and poles.– Again, let $f(z)$ be a doubly–periodic function with respect to Λ_τ. Let R be the fundamental parallelogram with vertices $0, 1, \tau, 1+\tau$ as in Figure 1 above. As we proved in the previous paragraph, there are only finitely many zeroes and poles of $f(z)$ within R. Therefore, by changing R to R_t with vertices $t, t+1, t+\tau$ and $t+1+\tau$, we can assume that none of the zeros/poles are on the boundary of R_t (see Figure 2 below).

Let $a_1, a_2, \ldots, a_k$ be the zeroes of $f(z)$ within R_t, of orders of vanishing $N_1, N_2, \ldots, N_k$ respectively. Similarly, let $b_1, b_2, \ldots, b_\ell$ be the poles of $f(z)$ within R_t, of orders $M_1, M_2, \ldots, M_\ell$ respectively (see Figure 3 below).

Theorem.

(1) $\displaystyle\sum_{j=1}^{\ell} \operatorname*{Res}_{z=b_j}(f(z)) = 0$. *In words, the sum of residues of $f(z)$ at poles within a fundamental parallelogram is zero.*

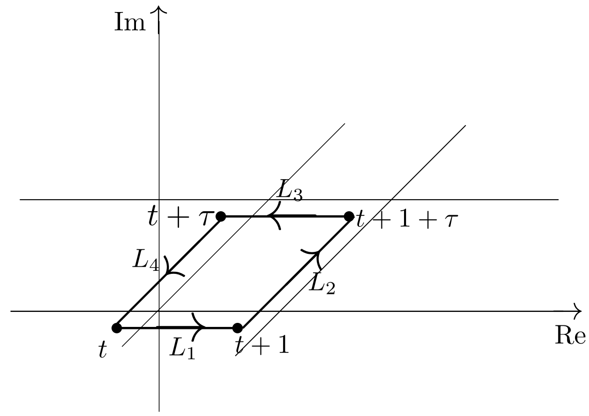

FIGURE 2. Fundamental parallelogram R_t has vertices $t, t+1, t+\tau, t+1+\tau$. $t \in \mathbb{C}$ is chosen so that $f(z)$ has no zeroes or poles on the boundary of R_t. C is the counterclockwise oriented boundary of R_t consisting of 4 smooth lines $L_1, \ldots, L_4$.

(2) $\sum_{i=1}^{k} N_i = \sum_{j=1}^{\ell} M_j$. *In words, the number of zeroes (counted with multiplicity) is equal to the number of poles (counted with multiplicity) within a fundamental parallelogram.*

(3) $\sum_{i=1}^{k} N_i a_i \equiv \sum_{j=1}^{\ell} M_j b_j$ *(modulo Λ_τ). In words, the sum of zeros is equal to the sum of poles plus an element of the form $m + n\tau$, where $m, n \in \mathbb{Z}$.*

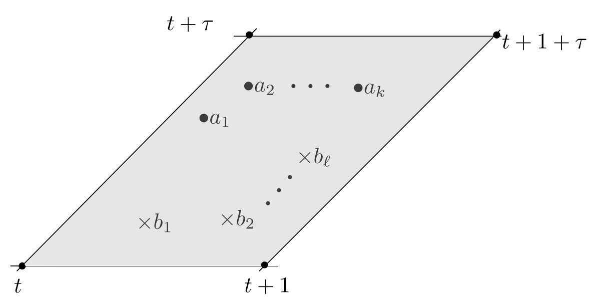

FIGURE 3. Zeroes $a_1, a_2, \ldots, a_k$, and Poles $b_1, b_2, \ldots, b_\ell$ of $f(z)$ within a fundamental parallelogram R_t.

Proof. Let C be the counterclockwise boundary of the fundamental parallelogram R_t (see Figure 2 above). It has four smooth pieces, all straight line segments L_1, L_2, L_3 and L_4.

Proof of (1). By Cauchy's residue theorem (see Lecture 26, §26.4, pages 3-4), we have:

$$\sum_{j=1}^{\ell} \operatorname*{Res}_{z=b_j}(f(z)) = \frac{1}{2\pi\mathbf{i}} \int_C f(z)\,dz$$

$$= \frac{1}{2\pi\mathbf{i}} \left(\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{L_3} f(z)\,dz + \int_{L_4} f(z)\,dz \right)$$

Note that $\int_{L_1} f(z)\,dz + \int_{L_3} f(z)\,dz = 0$ by periodicity of $f(z)$, as we demonstrate below:

$$\int_{L_3} f(z)\,dz = \int_{t+1+\tau}^{t+\tau} f(z)\,dz = -\int_{t+\tau}^{t+1+\tau} f(z)\,dz$$

Set $z = w + \tau$, to get (since $f(w+\tau) = f(w)$):

$$\int_{L_3} f(z)\,dz = -\int_{t}^{t+1} f(w+\tau)\,dw = -\int_{t}^{t+1} f(w)\,dw = -\int_{L_1} f(w)\,dw$$

Similarly, $\int_{L_2} f(z)\,dz + \int_{L_4} f(z)\,dz = 0$. Hence $\sum_{j=1}^{\ell} \operatorname*{Res}_{z=b_j}(f(z)) = 0$, as claimed.

Proof of (2). The number $\sum_{i=1}^{k} N_i - \sum_{j=1}^{\ell} M_j$ is computed by a similar contour integral (see Problem 12 of Set 7):

$$\sum_{i=1}^{k} N_i - \sum_{j=1}^{\ell} M_j = \frac{1}{2\pi\mathbf{i}} \int_C \frac{f'(z)}{f(z)}\,dz$$

$$= \frac{1}{2\pi\mathbf{i}} \left(\int_{L_1} \frac{f'(z)}{f(z)}\,dz + \int_{L_2} \frac{f'(z)}{f(z)}\,dz + \int_{L_3} \frac{f'(z)}{f(z)}\,dz + \int_{L_4} \frac{f'(z)}{f(z)}\,dz \right) .$$

By the exact same argument as in the proof of (1), we get that

$$\int_{L_1} \frac{f'(z)}{f(z)}\,dz + \int_{L_3} \frac{f'(z)}{f(z)}\,dz = 0 \text{ , and}$$

$$\int_{L_2} \frac{f'(z)}{f(z)}\,dz + \int_{L_4} \frac{f'(z)}{f(z)}\,dz = 0 .$$

This proves that $\sum_{i=1}^{k} N_i - \sum_{j=1}^{\ell} M_j = 0$.

Proof of (3). Again we realize the number $\sum_{i=1}^{k} N_i a_i - \sum_{j=1}^{\ell} M_j b_j$ as a contour integral:

$$\sum_{i=1}^{k} N_i a_i - \sum_{j=1}^{\ell} M_j b_j = \frac{1}{2\pi\mathbf{i}} \int_C \frac{z f'(z)}{f(z)}\,dz$$

(Proof of this equation is exactly the one you gave for Problem 5 of Homework 7 - it is therefore omitted here.)

Using the same argument as given for the proof of (1) above, we have:

$$\int_{L_1} \frac{z f'(z)}{f(z)}\,dz + \int_{L_3} \frac{z f'(z)}{f(z)}\,dz = \int_{t}^{t+1} \left(z\frac{f'(z)}{f(z)} - (z+\tau)\frac{f'(z+\tau)}{f(z+\tau)} \right) dz$$

$$= -\tau \int_{t}^{t+1} \frac{f'(z)}{f(z)}\,dz \qquad \text{(by periodicity of } f(z)\text{)}.$$

Similarly,

$$\int_{L_2} \frac{zf'(z)}{f(z)}\,dz + \int_{L_4} \frac{zf'(z)}{f(z)}\,dz = \int_t^{t+\tau} \frac{f'(z)}{f(z)}\,dz\ .$$

Combining these, we obtain:

$$\sum_{i=1}^{k} N_i a_i - \sum_{j=1}^{\ell} M_j b_j = \left(\frac{1}{2\pi\mathbf{i}} \int_t^{t+\tau} \frac{f'(z)}{f(z)}\,dz \right) - \tau \left(\frac{1}{2\pi\mathbf{i}} \int_t^{t+1} \frac{f'(z)}{f(z)}\,dz \right)\ .$$

The proof of (3) now boils down to showing that the two integrals written in the equation above are integers:

$$\frac{1}{2\pi\mathbf{i}} \int_t^{t+\tau} \frac{f'(z)}{f(z)}\,dz \in \mathbb{Z} \qquad \text{and} \qquad \frac{1}{2\pi\mathbf{i}} \int_t^{t+1} \frac{f'(z)}{f(z)}\,dz \in \mathbb{Z}\ .$$

Let us prove the first one (the proof of the second one being entirely analoguous). Consider the change of variables $w = f(z)$. It replaces the straight line $L : t \to t+\tau$ to a closed path γ starting at $\alpha = f(t)$ and ending at $f(t+\tau) = f(t) = \alpha$. Thus,

$$\frac{1}{2\pi\mathbf{i}} \int_t^{t+\tau} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi\mathbf{i}} \int_\gamma \frac{1}{w}\,dw$$

By Cauchy's integral formula, the last integral computes the number of times γ circles around 0 in counterclockwise fashion. Hence, it is an integer. Same argument applies to prove that $\frac{1}{2\pi\mathbf{i}} \int_t^{t+1} \frac{f'(z)}{f(z)}\,dz \in \mathbb{Z}$, and the theorem follows. □