COMPLEX ANALYSIS: LECTURE 34

(34.0) Review.– Let $\tau \in \mathbb{C}$ be such that $\text{Im}(\tau) > 0$. In the last lecture, we defined a *doubly–periodic function* with periods 1 and τ, as follows.

A meromorphic function $f : \mathbb{C} \dashrightarrow \mathbb{C}$ such that $f(z+1) = f(z)$ and $f(z+\tau) = f(z)$, is called a doubly–periodic function (with periods 1 and τ).

We introduced a notation $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z} \subset \mathbb{C}$. So, elements of Λ_τ are complex number of the form $m + n\tau$ where $m, n \in \mathbb{Z}$. Thus, any element of Λ_τ is a period of $f(z)$: $f(z + m + n\tau) = f(z)$ for every $m, n \in \mathbb{Z}$.

A *fundamental parallelogram* is any parallelogram R_t with vertices $\{t, t+1, t+\tau, t+1+\tau\}$ (here $t \in \mathbb{C}$ is an arbitrary complex number). Thus, we observe that the behaviour of a doubly–periodic function $f(z)$, $z \in \mathbb{C}$, is completely determined by $f(z^*)$, where z^* lies in R_t.

Being doubly–periodic is a very strong condition on a meromorphic function. We proved that any doubly–periodic function $f(z)$ must have the following properties:

(1) If $f(z)$ is holomorphic, then it is constant (see §33.2). In other words, if $f(z)$ is non–constant and doubly–periodic, then it must have some singularities, which have to be poles since $f(z)$ is assumed to be meromorphic. Similarly, it must also have zeroes - just run the same logic with $\dfrac{1}{f(z)}$.

(2) As is the property of meromorphic functions, they can only have finitely many poles in a closed and bounded subset of $\mathbb{C}$. Thus within a fundamental parallelogram R_t, $f(z)$ has only finitely many zeroes and poles (see §33.3). Of course, if $z = \alpha$ is a zero (or a pole) of $f(z)$, then so is any $\alpha + m + n\tau$, $m, n \in \mathbb{Z}$.

(3) Theorem 33.4 (1) shows that the sum of residues of $f(z)$ at poles within R_t has to be zero.

Let us list all the zeroes and poles of $f(z)$ within a fundamental parallelogram, according to their multiplicity (that is, if $z = \alpha$ is a zero of order 5, it must be listed 5 times):

$$a_1, a_2, \ldots, a_M : \text{ zeroes of } f(z) \text{ within } R_t.$$
$$b_1, b_2, \ldots, b_N : \text{ poles of } f(z) \text{ within } R_t.$$

(4) In Theorem 33.4 (2), we proved that $M = N$. That is, the number of zeroes is same as the number of poles (counted with multiplicity). Moreover, (Theorem 33.4 (3)):

$$\boxed{\sum_{i=1}^{N} a_i - \sum_{i=1}^{N} b_i \in \Lambda_\tau}$$

That is, sum of zeroes = sum of poles + an element of the form $m + n\tau$, where $m, n \in \mathbb{Z}$.

(34.1) Jacobi's theta function.– [1] Doubly–periodic functions are often expressed, and efficiently computed using *theta function*. Recall that $\tau \in \mathbb{C}$ is chosen such that $\mathrm{Im}(\tau) > 0$.

Definition. $\theta(z)$ is the unique holomorphic function, defined on the entire complex plane $\theta : \mathbb{C} \to \mathbb{C}$, which satisfies the following three properties:

(Periodicity) $\boxed{\theta(z+1) = -\theta(z)}$ and $\boxed{\theta(z+\tau) = -e^{-\pi \mathbf{i} \tau} e^{-2\pi \mathbf{i} z}\theta(z)}$

(Zeroes) $\boxed{\theta(z) = 0 \text{ if, and only if } z \in \Lambda_\tau}$

Moreover, the order of vanishing of $\theta(z)$ at the points $m + n\tau$ $(m, n \in \mathbb{Z})$ is 1.

(Normalization) $\boxed{\theta'(0) = 1}$

Remarks.

(1) If the dependence on the choice of τ needs to be highlighted, we will write $\theta(z; \tau)$ instead of just $\theta(z)$. As the notation indicates, sometimes people think of $\theta(z; \tau)$ as a function of two complex variables: let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half of the complex plane. Then, $\theta : \mathbb{C} \times \mathbb{H} \to \mathbb{C}$.

(2) Note that $\theta(z)$ is **not** doubly–periodic. This is a good news, since otherwise it cannot possibly be holomorphic, without being a boring constant.

(3) The definition above is incomplete. Namely, we still have to prove that such a function exists! It is done in Section 34.4 below, where a formula for $\theta(z)$ is written. However, it is remarkably easy to prove that if it exists, it is unique. The proposition below highlights this very useful idea - the three properties of $\theta(z)$ listed above are enough to prove all the identities involving $\theta(z)$.

(34.2) Uniqueness of $\theta(z)$.–

Proposition. *There can be at most one entire holomorphic function satisfying the three properties listed in Definition 34.1 above.*

[1] Carl Gustav Jacob Jacobi (1804-1851). *Fundamenta Nova Theoriae Functionum Ellipticarum, 1829.*

Proof. Let $f_1 : \mathbb{C} \to \mathbb{C}$ and $f_2 : \mathbb{C} \to \mathbb{C}$ be two holomorphic functions satisfying these properties. We want to prove that $f_1(z) = f_2(z)$ for every $z \in \mathbb{C}$.

Let $F(z) = \dfrac{f_1(z)}{f_2(z)}$. Then, we will first check that $F(z)$ is doubly–periodic, using the property called (Periodicity):

$$F(z+1) = \frac{f_1(z+1)}{f_2(z+1)} = \frac{-f_1(z)}{-f_2(z)} = F(z).$$

$$F(z+\tau) = \frac{f_1(z+\tau)}{f_2(z+\tau)} = \frac{-e^{-\pi\mathbf{i}\tau}e^{-2\pi\mathbf{i}z}f_1(z)}{-e^{-\pi\mathbf{i}\tau}e^{-2\pi\mathbf{i}z}f_2(z)} = \frac{f_1(z)}{f_2(z)} = F(z).$$

Next we show that $F(z)$ has no poles. The only points where $F(z)$ could possibly have poles are the zeroes of $f_2(z)$, namely $z = m + n\tau$, where $m, n \in \mathbb{Z}$ (we are using the property labelled (Zeroes) here). We can easily see that these are removable singularities of $F(z)$ (using l'hôpital rule, and (Normalization)):

$$\lim_{z\to 0} F(z) = \lim_{z\to 0}\frac{f_1(z)}{f_2(z)} = \lim_{z\to 0}\frac{f_1'(z)}{f_2'(z)} = \frac{f_1'(0)}{f_2'(0)} = 1 \text{ exists.}$$

Thus, $z = 0$ is not a pole of $F(z)$. By its double–periodicity, $z = m + n\tau$ is not a pole of $F(z)$ for any $m, n \in \mathbb{Z}$. Hence, $F(z)$ is holomorphic and doubly–periodic. By Theorem 33.2, $F(z)$ has to be a constant, say $F(z) = C$ for every $z \in \mathbb{C}$. This constant has already been computed above: $C = F(0) = 1$. Hence $f_1(z) = f_2(z)$, as claimed. □

(34.3) An infinite product.– Consider the following infinite product

$$\theta^+(z) = \prod_{n=1}^{\infty}\left(1 - e^{2\pi\mathbf{i}n\tau}e^{2\pi\mathbf{i}z}\right)$$

Theorem. *This infinite product converges uniformly on compact subsets of* $\mathbb{C}$. *Hence,* $\theta^+ : \mathbb{C} \to \mathbb{C}$ *is a holomorphic function. It satisfies the following properties:*

(1) $\theta^+(z+1) = \theta^+(z)$, *and*

$$\theta^+(z+\tau) = \frac{\theta^+(z)}{1 - e^{2\pi\mathbf{i}\tau}e^{2\pi\mathbf{i}z}} .$$

(2) $\theta^+(z) = 0$ *if, and only if* $z = m - n\tau$, *where* $m \in \mathbb{Z}$ *and* $n \in \mathbb{Z}_{\geq 1}$. *All these zeroes of* $\theta^+(z)$ *are of order* 1.

Proof. Let us begin by proving the convergence [2]. This proof is very similar to the one given in Lecture 32, §32.2. For the purposes of this proof, let us denote $p = e^{2\pi\mathbf{i}\tau}$. Since $\mathrm{Im}(\tau) > 0$, we get that $|p| = e^{-2\pi\,\mathrm{Im}(\tau)} < 1$.

Let $D \subset \mathbb{C}$ be a compact subset. Choose $A \in \mathbb{R}$ such that $\mathrm{Im}(z) > A$ for every $z \in D$ (see Figure 1 below).

[2]Optional. Proofs of (1) and (2) are not optional, being really easy.

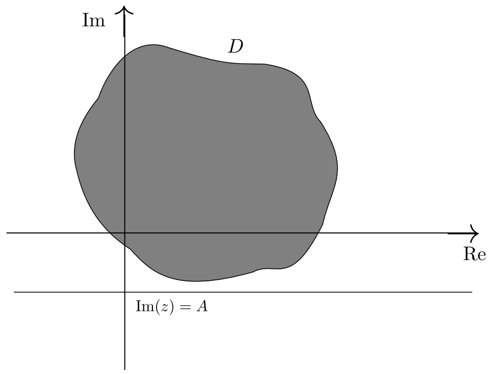

FIGURE 1. Given a compact set D, we can choose $A \in \mathbb{R}$ such that $\mathrm{Im}(z) > A$ for every $z \in D$.

Since $|p| < 1$, we know that $|p|^n \to 0$ as $n \to \infty$. Therefore, we can pick N_0 large enough so that

$$|p^N| < \frac{1}{2} \cdot e^{2\pi A} \text{ for every } N \geq N_0.$$

This implies that, for every $z \in D$ and $N \geq N_0$, we have: $\left|p^N e^{2\pi \mathbf{i} z}\right| = |p|^N e^{-2\pi \operatorname{Im}(z)} \leq |p|^N e^{-2\pi A} < \frac{1}{2}$. Thus $\ln\left(1 - p^N e^{2\pi \mathbf{i} z}\right)$ can be expanded using its Taylor series, and its modulus can be bounded using the triangle inequality as follows.

$$\begin{aligned} \left|\ln\left(1 - p^N e^{2\pi \mathbf{i} z}\right)\right| &= \left|p^N e^{2\pi \mathbf{i} z} + \frac{1}{2} p^{2N} e^{4\pi \mathbf{i} z} + \frac{1}{3} p^{3N} e^{6\pi \mathbf{i} z} \cdots\right| \\ &\leq |p|^N |e^{2\pi \mathbf{i} z}| \left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right) \leq 2e^{-2\pi A} |p|^N. \end{aligned}$$

$$\Rightarrow \left|\sum_{N=N_0}^{\infty} \ln\left(1 - p^N e^{2\pi \mathbf{i} z}\right)\right| \leq 2e^{-2\pi A} \sum_{N=N_0}^{\infty} |p|^N \text{ is finite, since } |p| < 1.$$

Hence, $\sum_{N=N_0}^{\infty} \ln\left(1 - p^N e^{2\pi \mathbf{i} z}\right)$ converges uniformly in $z \in D$. This proves the theorem, since:

$$\theta^+(z) = \prod_{n=1}^{N_0 - 1} (1 - p^n e^{2\pi \mathbf{i} z}) \cdot \exp\left(\sum_{N=N_0}^{\infty} \ln\left(1 - p^N e^{2\pi \mathbf{i} z}\right)\right).$$

Proof of (1). Since $\theta^+(z)$ only involves $e^{2\pi \mathbf{i} z}$ which is periodic in $z \mapsto z + 1$, we get that $\theta^+(z+1) = \theta^+(z)$. Moreover,

$$\theta^+(z + \tau) = \prod_{n=1}^{\infty} \left(1 - e^{2\pi \mathbf{i}(n+1)\tau} e^{2\pi \mathbf{i} z}\right) = \frac{\theta^+(z)}{1 - e^{2\pi \mathbf{i} \tau} e^{2\pi \mathbf{i} z}}.$$

Proof of (2). Since $\theta^+(z)$ is given as a product, we have $\theta^+(z) = 0 \iff 1 - e^{2\pi \mathbf{i} n \tau} e^{2\pi \mathbf{i} z} = 0$, for some $n \geq 1$. That is:

$$e^{2\pi \mathbf{i}(z + n\tau)} = 1 \iff z + n\tau = m \text{ for some } m \in \mathbb{Z}.$$

This is same as saying that $z = m - n\tau$, where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 1}$. Since all the zeroes of $1 - e^{2\pi \mathbf{i}(z+n\tau)}$ are of order 1, the same is true for $\theta^+(z)$. □

(34.4) Existence of $\theta(z)$**.–** Let $\theta^+(z)$ be the holomorphic entire function from the previous paragraph. Let us define $\theta^-(z) = \theta^+(-z)$, that is, $\theta^-(z) = \prod_{n=1}^{\infty}\left(1 - e^{2\pi \mathbf{i} n\tau}e^{-2\pi \mathbf{i} z}\right)$. Analoguous to the two properties of $\theta^+(z)$ from Theorem 34.3 above, we have:

(1) $\theta^-(z+1) = \theta^-(z)$ and
$$\theta^-(z+\tau) = \left(1 - e^{-2\pi \mathbf{i} z}\right)\theta^-(z).$$

(2) $\theta^-(z) = 0$ if, and only if $z = m + n\tau$ where $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_{\geq 1}$. These zeroes of $\theta^-(z)$ are of order 1.

Define: $\boxed{T(z) = \sin(\pi z)\theta^+(z)\theta^-(z)}$ We are going to prove that $T(z)$ satisfies the first two of the three properties listed in the definition of the theta function in §34.1.

Proof of (Periodicity): Since $\theta^{\pm}(z+1) = \theta^{\pm}(z)$ and $\sin(\pi(z+1)) = -\sin(\pi z)$, we immediately get: $T(z+1) = -T(z)$.

Writing $\sin(\pi z) = \dfrac{e^{\pi \mathbf{i} z} - e^{-\pi \mathbf{i} z}}{2\mathbf{i}}$, and using the periodicities of $\theta^{\pm}(z)$ established above, we can carry out the following computation:
$$\frac{T(z+\tau)}{T(z)} = \frac{e^{\pi \mathbf{i}(z+\tau)} - e^{-\pi \mathbf{i}(z+\tau)}}{e^{\pi \mathbf{i} z} - e^{-\pi \mathbf{i} z}} \cdot \frac{1 - e^{-2\pi \mathbf{i} z}}{1 - e^{2\pi \mathbf{i}(z+\tau)}} = -e^{-\pi \mathbf{i}\tau}e^{-2\pi \mathbf{i} z}.$$

Proof of (Zeroes): $T(z) = 0$ if, and only if
$$\text{Either } \sin(\pi z) = 0, \quad \text{or } \theta^+(z) = 0, \quad \text{or } \theta^-(z) = 0.$$
That is, $z \in \mathbb{Z}$, or $z \in \mathbb{Z} - \tau\mathbb{Z}_{\geq 1}$, or $z \in \mathbb{Z} + \tau\mathbb{Z}_{\geq 1}$. Moreover, all these zeroes are of order 1. Hence, $T(z) = 0 \iff z \in \mathbb{Z} + \tau\mathbb{Z}$.

It remains to normalize $T(z)$ so that the third property also holds. This simply means that we have to divide $T(z)$ by $T'(0)$ which will give us the theta function.
$$T'(z) = \pi\cos(\pi z)(\theta^+(z)\theta^-(z)) + \sin(\pi z)(\theta^+(z)\theta^-(z))'$$
$$\Rightarrow T'(0) = \pi\theta^+(0)\theta^-(0) = \pi\prod_{n=1}^{\infty}\left(1 - e^{2\pi \mathbf{i} n\tau}\right)^2.$$

Hence, $\theta(z) = \dfrac{T(z)}{T'(0)}$. Explicitly, we have the following product formula for $\theta(z)$:
$$\boxed{\theta(z) = \frac{\sin(\pi z)}{\pi} \cdot \frac{\theta^+(z)\theta^-(z)}{\theta^+(0)\theta^-(0)}}$$

Unraveling the definitions of $\theta^{\pm}(z)$, the product formula of $\theta(z)$ takes the following form:

$$\boxed{\theta(z) = \frac{e^{\pi \mathbf{i} z} - e^{-\pi \mathbf{i} z}}{2\pi \mathbf{i}} \cdot \prod_{n=1}^{\infty} \frac{\left(1 - e^{2\pi \mathbf{i} n\tau} e^{2\pi \mathbf{i} z}\right)\left(1 - e^{2\pi \mathbf{i} n\tau} e^{-2\pi \mathbf{i} z}\right)}{\left(1 - e^{2\pi \mathbf{i} n\tau}\right)^2}}$$

Corollary. *Theta function is odd:* $\boxed{\theta(-z) = -\theta(z)}$.

(34.5) Doubly–periodic functions in terms of $\theta(z)$.– We are now going to prove that any doubly–periodic function can be written in terms of Jacobi's theta function.

Theorem. *Let $a_1, a_2, \ldots, a_N \in \mathbb{C}$ and $b_1, b_2, \ldots, b_N \in \mathbb{C}$ be two collections of complex numbers* [3] *such that:*

$$a_1 + a_2 + \cdots + a_N = b_1 + b_2 + \cdots + b_N$$

Then $f(z) = \prod_{k=1}^{N} \frac{\theta(z - a_k)}{\theta(z - b_k)}$ is a doubly–periodic function. Moreover, any doubly–periodic function can be written in this form, up to multiplication by a constant.

Proof. We begin by proving that the function $f(z) = \prod_{k=1}^{N} \frac{\theta(z - a_k)}{\theta(z - b_k)}$ is doubly–periodic. This is easily proven using the periodicity property of theta function, see §34.1 above. It is clear that $f(z + 1) = f(z)$. Now, we have:

$$f(z+\tau) = \prod_{k=1}^{N} \frac{\theta(z + \tau - a_k)}{\theta(z + \tau - b_k)} = \prod_{k=1}^{N} \left(\frac{-e^{-\pi \mathbf{i}\tau} e^{-2\pi \mathbf{i}(z - a_k)}}{-e^{-\pi \mathbf{i}\tau} e^{-2\pi \mathbf{i}(z - b_k)}} \cdot \frac{\theta(z - a_k)}{\theta(z - b_k)} \right)$$

$$= e^{2\pi \mathbf{i}\left(\sum_{k=1}^{N} a_k - \sum_{k=1}^{N} b_k\right)} \frac{\theta(z - a_k)}{\theta(z - b_k)} = \frac{\theta(z - a_k)}{\theta(z - b_k)} = f(z).$$

Now let $g(z)$ be an arbitrary doubly–periodic function. Let us list its zeroes and poles within a fundamental parallelogram, according to their multiplicity (see §34.0 above). Let $x_1, x_2, \ldots, x_M$ be the zeroes and $y_1, y_2, \ldots, y_M$ be the poles. By Theorem 33.4 (3):

$$(x_1 + x_2 + \cdots + x_M) = (y_1 + y_2 + \cdots + y_M) + m + n\tau \ ,$$

for some $m, n \in \mathbb{Z}$. Replace y_M by $y_M + m + n\tau$ so that $\sum_{k=1}^{M} x_k = \sum_{k=1}^{M} y_k$. By the argument given above, we have that:

$$g_1(z) = \prod_{k=1}^{M} \frac{\theta(z - x_k)}{\theta(z - y_k)} \text{ is doubly–periodic.}$$

Moreover $g(z)/g_1(z)$ does not have any poles (check this by yourself - zeroes and poles of $g(z)$ and $g_1(z)$ cancel each other). Hence, by Theorem 33.2, it must be a constant $C \in \mathbb{C}$.

[3] These numbers are not assumed to be distinct - repetitions are allowed.

Therefore, we have proven that an arbitrary doubly–periodic function $g(z)$ can be written as:

$$g(z) = C \prod_{k=1}^{M} \frac{\theta(z - x_k)}{\theta(z - y_k)} \ .$$

□

Example. A lot of examples of doubly–periodic functions can now be given. For instance, pick $a \in \mathbb{C}$ such that $2a \notin \Lambda_\tau$. Then the following function is doubly–periodic:

$$\frac{\theta(z + a)\theta(z - a)}{\theta(z)^2}.$$

It has a pole of order 2 at every $z = m + n\tau$ $(m, n \in \mathbb{Z})$. It has zeroes of order 1 at $z = \pm a + m + n\tau$.

If, contrary to our assumption, $2a$ does belong to Λ_τ, then this function would have zeroes of order 2 (since in this case $a = -a + m + n\tau$). Even further, if the complex number a we picked is from Λ_τ, then we can use the periodicity of $\theta(z)$ to conclude that the function written above is just a constant.