COMPLEX ANALYSIS: LECTURE 35

(35.0) Review.– Let $\tau \in \mathbb{C}$ be such that $\mathrm{Im}(\tau) > 0$. We defined Jacobi's theta function, denoted by $\theta(z)$ (or, by $\theta(z;\tau)$ if dependence on τ needs to be highlighted) in Lecture 34, §34.1, as follows.

$\theta(z)$ is the unique holomorphic function, defined on the entire complex plane $\theta : \mathbb{C} \to \mathbb{C}$, which satisfies the following three properties:

(Periodicity) $$\theta(z+1) = -\theta(z) \quad \text{and} \quad \theta(z+\tau) = -e^{-\pi \mathbf{i}\tau} e^{-2\pi \mathbf{i} z}\theta(z)$$

(Zeroes) $$\theta(z) = 0 \text{ if, and only if } z \in \Lambda_\tau$$

Moreover, the order of vanishing of $\theta(z)$ at the points $m + n\tau$ $(m, n \in \mathbb{Z})$ is 1.

(Normalization) $$\theta'(0) = 1$$

(1) An infinite product expression for $\theta(z)$ was given in Lecture 34, §34.4. From this we concluded that $\theta(-z) = -\theta(z)$.

(2) We proved in Lecture 34, §34.2 that the three properties listed above uniquely determine $\theta(z)$.

(3) Finally, we showed that any doubly–periodic function can be written in terms of theta function. More precisely, if $f(z)$ is a doubly–periodic function, then we can find $a_1, a_2, \ldots, a_N \in \mathbb{C}$ and $b_1, b_2, \ldots, b_N \in \mathbb{C}$ two collections of complex numbers satisfying
$$a_1 + a_2 + \cdots + a_N = b_1 + b_2 + \cdots + b_N \ ,$$
such that $f(z) = C \displaystyle\prod_{k=1}^{N} \frac{\theta(z-a_k)}{\theta(z-b_k)}$, where $C \in \mathbb{C}$ is a constant.

A word on the method of proofs. In Lecture 34, Proposition 34.2 and Theorem 34.5 are proved using a very simple and elegant argument, which I would like to emphasize here. Assume that we want to prove an identity of the form $f_1(z) = f_2(z)$. We can do this in the following three steps:

- Prove that $F(z) = \dfrac{f_1(z)}{f_2(z)}$ is doubly–periodic.
- Prove that $F(z)$ is holomorphic, by showing that zeroes and poles of f_1 and f_2 cancel each other. At this point we invoke Theorem 33.2 to conclude that $F(z) = C$ is a constant.
- Finally determine C by evaluating $F(z)$ at a convenient point $z = z_0$.

(35.1) What is and isn't in these notes.– We will illustrate the method sketched above, by proving a non–trivial identity (often called *Fay's trisecant identity* [1]) involving theta function in §35.2 below. In general, it is the method that is important, and any such identity can be easily proved using it. But I will still write a nice way to memorize Fay's identity, in §35.3. It has been conjectured that any relation involving theta function can be deduced from Fay's trisecant identity [2] but as far as I know, there is no proof of this.

In §35.4, I will illustrate a systemetic way of obtaining a formula of an infinite sum which satisfies the exact same periodicity property as $\theta(z)$. The resulting infinite sum is shown to be uniformly convergent in §35.5. Therefore, it defines an entire holomorphic function, denoted by $\theta_1(z)$. In Theorem 35.5, we will show that $\theta_1(z)$ has zeroes of order 1 at $z = m+n\tau$ where $m, n \in \mathbb{Z}$. Hence, $\theta_1(z)$ and $\theta(z)$ must be related by a constant[3].

The computation of this constant is **not** in these notes, though its value is given in §35.6. It relies on a rather non–trivial identity called *Jacobi's triple product identity* (because it features three infinite products):

$$\boxed{\left(\prod_{n=1}^{\infty}\left(1-e^{2\pi \mathbf{i} n\tau}\right)\right)\cdot\left(\prod_{n=1}^{\infty}\left(1-e^{2\pi \mathbf{i} n\tau}e^{2\pi \mathbf{i} z}\right)\right)\cdot\left(\prod_{n=1}^{\infty}\left(1-e^{2\pi \mathbf{i} n\tau}e^{-2\pi \mathbf{i} z}\right)\right) = \sum_{k=0}^{\infty}(-1)^k e^{\pi \mathbf{i}\tau k(k+1)}\frac{\sin((2k+1)\pi z)}{\sin(\pi z)}}$$

This identity appeared in Jacobi's *Fundamenta Nova Theoriae Functionum Ellipticarum (1829)*, and is one of the most beautiful formulae in the theory of doubly–periodic functions. It gives rise to several interesting combinatorial results. For instance, by letting $z \to 0$, and writing $q = e^{\pi \mathbf{i}\tau}$, we get:

$$\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)} = \prod_{n=1}^{\infty}(1-q^{2n})^3$$

In turn, such combinatorial formulae go back to Euler (1775). I encourage you to read more about these, for instance search for *Euler's pentagonal number theorem.*

On the proofs of Jacobi's triple product identity. I know two ways to prove this, none are elementary. One of these proofs will be written up as Optional Reading B, and is the same as Jacobi's original proof. It involves rather lengthy, but still elementary, computations with the heat equation (a differential equation satisfied by theta function).

I.G. Macdonald obtained a generalization of this identity, now known as Macdonald identities, in *Affine root systems and Dedekind's η function (Invent. Math. 1972)*. It was later realized by V.G. Kac (1974) and R.V. Moody (1975) that all of these identities can be obtained from the *Weyl–Kac character formula*. This is the second proof I know, but I will

[1] J. Fay: *Theta functions on Riemann surfaces (1973).*

[2] D. Mumford: *Tata lectures on theta II (1984).*

[3] Historically, it is $\theta_1(z)$ that appeared first in the works of Jacobi.

leave it for you to learn it in a course on representation theory.

(35.2) Fay's trisecant identity.– This identity is a relation between values of theta function at sums and differences of 4 complex numbers. It only looks scary, but is very easy to remember (see §35.3 below) and prove.

Theorem. *Let $\alpha, \beta, \gamma, \delta$ be four complex numbers. Then:*

$$\theta(\alpha-\gamma)\theta(\alpha+\gamma)\theta(\beta-\delta)\theta(\beta+\delta) = \theta(\alpha-\beta)\theta(\alpha+\beta)\theta(\gamma-\delta)\theta(\gamma+\delta) + \theta(\alpha-\delta)\theta(\alpha+\delta)\theta(\beta-\gamma)\theta(\beta+\gamma)$$

Proof. Let us keep $\beta, \gamma, \delta \in \mathbb{C}$ fixed and view both sides as functions of $z = \alpha$. Let us consider the following function:

$$F(z) = \frac{\theta(z-\beta)\theta(z+\beta)\theta(\gamma-\delta)\theta(\gamma+\delta) + \theta(z-\delta)\theta(z+\delta)\theta(\beta-\gamma)\theta(\beta+\gamma)}{\theta(z-\gamma)\theta(z+\gamma)}$$

We want to show that $F(z)$ is a constant, given by $\theta(\beta-\delta)\theta(\beta+\delta)$. To do this, we proceed with the method outlined in §35.0 above.

Step 1. Prove that $F(z)$ is doubly–periodic.

It is easy to check that $F(z+1) = F(z)$. Now, using $\theta(x+\tau) = -e^{-\pi \mathbf{i}\tau}e^{-2\pi\mathbf{i}x}\theta(x)$, we get:

$$\theta(z+\tau-\gamma)\theta(z+\tau+\gamma) = e^{-2\pi\mathbf{i}\tau}e^{-4\pi\mathbf{i}z}\theta(z-\gamma)\theta(z+\gamma).$$

(same with γ changed to β or δ).

Thus, upon replacing z by $z+\tau$, both the numerator and the denominator of $F(z)$ get rescaled by $e^{-2\pi\mathbf{i}\tau}e^{-4\pi\mathbf{i}z}$. Hence we get $F(z+\tau) = F(z)$.

Step 2. Prove that $F(z)$ is holomorphic.

The apparent poles of $F(z)$ are at the zeroes of $\theta(z-\gamma)\theta(z+\gamma)$. That is, at $z = \pm\gamma + m + n\tau$, where $m, n \in \mathbb{Z}$. By periodicity of $F(z)$ already established, it is enough to show that $z = \pm\gamma$ are removable poles of $F(z)$. Since the order of vanishing of $\theta(z-\gamma)$ at $z = \gamma$ is 1, we only have to check that the numerator of $F(z)$ also vanishes at $z = \gamma$ (same argument applies to $z = -\gamma$ as well). Let us set $z = \gamma$ in the numerator of $F(z)$, and use the fact that $\theta(-x) = -\theta(x)$:

$$\begin{aligned}\theta(\gamma-\beta)\theta(\gamma+\beta)\theta(\gamma-\delta)\theta(\gamma+\delta) + \theta(\gamma-\delta)\theta(\gamma+\delta)\theta(\beta-\gamma)\theta(\beta+\gamma) \\ = \theta(\gamma-\delta)\theta(\gamma+\delta)\theta(\beta+\gamma)\left(\theta(-(\beta-\gamma)) + \theta(\beta-\gamma)\right) \\ = 0.\end{aligned}$$

So, $F(z)$ is holomorphic and doubly–periodic, hence it must be a constant $F(z) = C$. The last step is to figure out the constant, which we can by specializing z to a convenient value. For instance, let us set $z = \beta$:

$$C = F(\beta) = \theta(\beta-\delta)\theta(\beta+\delta).$$

The theorem is proved. □

(35.3) How to memorize Fay's identity.– In year 130, Greek mathematician Ptolemy proved the following result (called Ptolemy relation). Let $ABCD$ be a quadrilateral inscribed in a circle (see Figure 1 below). Denoting by $|PQ|$ the length of the line segment joining two points P and Q, the following relation must hold.

(Ptolemy relation) $$\boxed{|AC|\cdot|BD| = |AB|\cdot|CD| + |AD|\cdot|BC|}$$

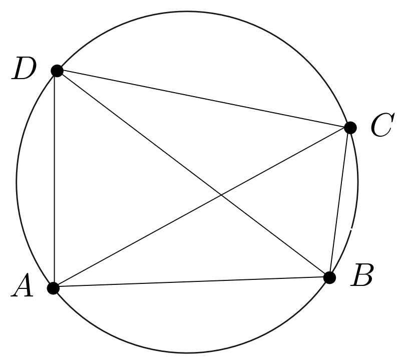

FIGURE 1. A quadrilateral $ABCD$ inscribed in a circle.

There are several ways to prove this relation (using similar triangles, trigonometric identities etc). I will omit a proof of this, since this section is more of a memorization trick.

To get Fay's trisecant identity, simply replace

$$|AC| \mapsto \theta(\alpha-\gamma)\theta(\alpha+\gamma), \qquad |BD| \mapsto \theta(\beta-\delta)\theta(\beta+\delta)$$

and similarly for other sides. Upon this replacement, Ptolemy relation becomes Fay's trisecant identity.

(35.4) Theta function as a Fourier series.– Let us consider the problem of finding a function, say $\theta_1(z)$, which has the same periodicity property as $\theta(z)$:

$$\theta_1(z+1) = -\theta_1(z) \qquad \text{and} \qquad \theta_1(z+\tau) = -e^{-\pi \mathbf{i}\tau}e^{-2\pi \mathbf{i}z}\theta_1(z).$$

From the first equation, we can guess that the function we are looking for can be written as an infinite series in $e^{\pi\mathbf{i}(2\ell+1)z}$, where $\ell\in\mathbb{Z}$. This is because $e^{\pi\mathbf{i}(2\ell+1)} = -1$ for every $\ell\in\mathbb{Z}$. So we postulate:

$$\theta_1(z) = \sum_{\ell\in\mathbb{Z}} c_\ell e^{\pi\mathbf{i}(2\ell+1)z} \ .$$

In order to figure out the coefficients (c_ℓ, $\ell\in\mathbb{Z}$), we use the second periodicity requirement.

$$\theta_1(z+\tau) = \sum_{\ell\in\mathbb{Z}} c_\ell e^{\pi\mathbf{i}(2\ell+1)\tau}e^{\pi\mathbf{i}(2\ell+1)z}$$

$$-e^{-\pi\mathbf{i}\tau}e^{-2\pi\mathbf{i}z}\theta_1(z) = \sum_{\ell\in\mathbb{Z}}(-c_\ell e^{-\pi\mathbf{i}\tau})e^{\pi\mathbf{i}(2\ell-1)z} = \sum_{n\in\mathbb{Z}}(-c_{n+1}e^{-\pi\mathbf{i}\tau})e^{\pi\mathbf{i}(2n+1)z}$$

where, in the last step, I substituted $\ell = n+1$. Now comparing the coefficients of $e^{2\pi\mathbf{i}(2\ell+1)z}$, we obtain:

$$-c_{\ell+1}e^{-\pi\mathbf{i}\tau} = c_\ell e^{\pi\mathbf{i}(2\ell+1)\tau} \Rightarrow \boxed{c_{\ell+1} = -c_\ell e^{\pi\mathbf{i}(2\ell+2)\tau}}$$

Let us agree to normalize $c_0 = 1$, so that the above relation can be solved as:

$$c_\ell = (-1)^\ell e^{\pi\mathbf{i}\tau\ell(\ell+1)}.$$

This computation motivates us to define a function:

$$\boxed{\theta_1(z) = \sum_{\ell\in\mathbb{Z}}(-1)^\ell e^{\pi\mathbf{i}\tau\ell(\ell+1)}e^{\pi\mathbf{i}(2\ell+1)z}}$$

(35.5) Convergence and zeroes of $\theta_1(z)$.– As always, our first task is going to be to prove that the infinite sum written above converges uniformly on compact subsets of $\mathbb{C}$, hence defines $\theta_1(z)$ as a holomorphic function $\theta_1 : \mathbb{C} \to \mathbb{C}$. We will also have to know the zeroes of $\theta_1(z)$.

In order to do this, it is convenient to group certain terms of $\theta_1(z)$ together. Note that the coefficient $e^{\pi\mathbf{i}\tau\ell(\ell+1)}$ gives the same value for $\ell = k$ and $\ell = -k-1$, where $k \in \mathbb{Z}_{\geq 0}$. This allows us to write $\theta_1(z)$ as an infinite sum over $k \in \mathbb{Z}_{\geq 0}$, combining terms $(0,-1)$, $(1,-2)$, and so on.

$$\boxed{\theta_1(z) = \sum_{k=0}^{\infty}(-1)^k e^{\pi\mathbf{i}\tau k(k+1)}\left(e^{\pi\mathbf{i}(2k+1)z} - e^{-\pi\mathbf{i}(2k+1)z}\right)}$$

Theorem. *The infinite sum given above converges uniformly on compact subsets of* $\mathbb{C}$. *Thus, $\theta_1(z)$ is an entire holomorphic function with the following periodicity properties:*

$$\theta_1(z+1) = -\theta_1(z) \qquad \textit{and} \qquad \theta_1(z+\tau) = -e^{-\pi\mathbf{i}\tau}e^{-2\pi\mathbf{i}z}\theta_1(z).$$

Moreover, $\theta_1(z) = 0$ if, and only if $z = m + n\tau$, where $m, n \in \mathbb{Z}$. Each of these zeroes is of order 1.

Proof. We begin by proving the uniform convergence [4]. Let $D \subset \mathbb{C}$ be a compact subset. Choose $A \in \mathbb{R}_{>0}$ so that $-A < \mathrm{Im}(z) < A$ for every $z \in D$. Let us write $q = e^{\pi\mathbf{i}\tau}$. Note that $|q| = e^{-\pi\,\mathrm{Im}(\tau)} < 1$ since $\mathrm{Im}(\tau) > 0$.

So, for every $z \in D$, and $k \in \mathbb{Z}_{\geq 0}$, we have:

$$\left|e^{\pi\mathbf{i}(2k+1)z}\right| = e^{-\pi(2k+1)\,\mathrm{Im}(z)} < \left(e^{\pi A}\right)^{2k+1}$$

and similarly $\left|e^{-\pi\mathbf{i}(2k+1)z}\right| < \left(e^{\pi A}\right)^{2k+1}$. Therefore we obtain:

$$\left|e^{\pi\mathbf{i}(2k+1)z} - e^{-\pi\mathbf{i}(2k+1)z}\right| < 2\left(e^{\pi A}\right)^{2k+1}.$$

[4] This part of the proof is optional.

Now, the ratio test implies that the following series, which dominates the series defining $\theta_1(z)$, converges:

$$2\sum_{k=0}^{\infty} |q|^{k(k+1)} \left(e^{\pi A}\right)^{2k+1}$$

Ratio of successive terms is $|q|^{2(k+1)}e^{2\pi A} \to 0$ as $k \to \infty$, since $|q| < 1$.

Hence, the infinite series defining $\theta_1(z)$ converges uniformly in $z \in D$, as we wanted.

The periodicity properties of $\theta_1(z)$ have already been verified in §35.4 above. It is also easy to see that $\theta_1(0) = 0$ from the formula written above. By periodicity, we conclude that $\theta_1(m + n\tau) = 0$ for every $m, n \in \mathbb{Z}$. Thus, it remains to show that these are the only zeroes of θ_1, and the order of vanishing of $\theta_1(z)$ at $z = 0$ is 1.

Let us compute the number of zeroes of $\theta_1(z)$ within a fundamental parallelogram R_t with vertices $t, t+1, t+\tau, t+1+\tau$ (see Figure 2 below).

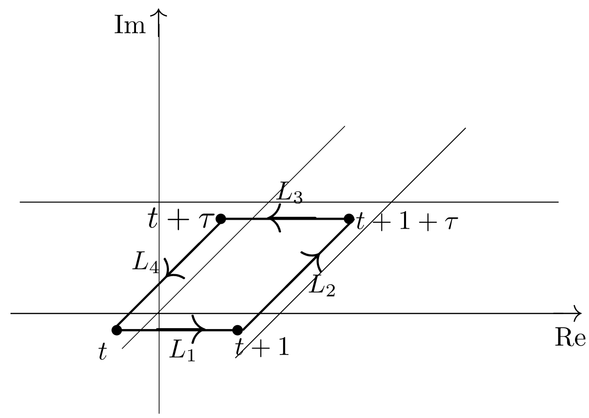

FIGURE 2. Contour C is the counterclockwise boundary of the parallelogram R_t.

As we know by now (Problem 5 of Homework 7), if N is the number of zeroes of $\theta_1(z)$ within C, then: $N = \dfrac{1}{2\pi \mathbf{i}} \displaystyle\int_C \frac{\theta_1'(z)}{\theta_1(z)}\, dz$. This integral computes number of zeroes - number of poles, but we already know $\theta_1(z)$ is holomorphic on $\mathbb{C}$, so it does not have any poles.

$$N = \frac{1}{2\pi \mathbf{i}}\left(\int_{L_1} \frac{\theta_1'(z)}{\theta_1(z)}\, dz + \int_{L_2} \frac{\theta_1'(z)}{\theta_1(z)}\, dz + \int_{L_3} \frac{\theta_1'(z)}{\theta_1(z)}\, dz + \int_{L_4} \frac{\theta_1'(z)}{\theta_1(z)}\, dz\right)$$

Using the periodicity properties of $\theta_1(z)$, we can compute:

$$\frac{\theta_1'(z+1)}{\theta_1(z+1)} = \frac{\theta_1'(z)}{\theta_1(z)} \qquad \text{and} \qquad \frac{\theta_1'(z+\tau)}{\theta_1(z+\tau)} = -2\pi\mathbf{i} + \frac{\theta_1'(z)}{\theta_1(z)}$$

Hence the integral over L_2 cancels with the integral over L_4. Moreover,

$$\int_{L_3} \frac{\theta_1'(z)}{\theta_1(z)}\, dz = -\int_{t+\tau}^{t+1+\tau} \frac{\theta_1'(z)}{\theta_1(z)}\, dz = -\int_t^{t+1} \frac{\theta_1'(z+\tau)}{\theta_1(z+\tau)}\, dz$$

$$= -\int_t^{t+1} \left(-2\pi\mathbf{i} + \frac{\theta_1'(z)}{\theta_1(z)}\right) dz = -\int_{L_1} \frac{\theta_1'(z)}{\theta_1(z)}\, dz + 2\pi\mathbf{i}.$$

This implies that

$$\frac{1}{2\pi\mathbf{i}} \int_C \frac{\theta_1'(z)}{\theta_1(z)}\, dz = 1.$$

Since we already know $\theta_1(0) = 0$, and there is only one zero of $\theta_1(z)$ within C, it must be at $z = 0$, with order of vanishing 1, and nowhere else. The theorem is proved. □

(35.6) $\theta_1(z)$ **vs** $\theta(z)$**.**– In the previous two paragraphs, we proved that

$$\theta_1(z) = \sum_{k=0}^{\infty} (-1)^k e^{\pi\mathbf{i}\tau k(k+1)} \left(e^{\pi\mathbf{i}(2k+1)z} - e^{-\pi\mathbf{i}(2k+1)z}\right)$$

satisfies two of the three defining properties of

$$\theta(z) = \frac{e^{\pi\mathbf{i}z} - e^{-\pi\mathbf{i}z}}{2\pi\mathbf{i}} \cdot \prod_{n=1}^{\infty} \frac{\left(1 - e^{2\pi\mathbf{i}n\tau} e^{2\pi\mathbf{i}z}\right)\left(1 - e^{2\pi\mathbf{i}n\tau} e^{-2\pi\mathbf{i}z}\right)}{\left(1 - e^{2\pi\mathbf{i}n\tau}\right)^2}.$$

Hence, on general grounds (see Proposition 34.2 of Lecture 34) they must be related by a constant, say C (which depends on τ, but is independent of z): $\theta_1(z) = C\theta(z)$. Jacobi's triple product identity [5] computes this constant as:

$$C = 2\pi\mathbf{i} \prod_{n=1}^{\infty} \left(1 - e^{2\pi\mathbf{i}n\tau}\right)^3$$

Combining all of this, and writing $\dfrac{e^{\pi\mathbf{i}(2k+1)z} - e^{-\pi\mathbf{i}(2k+1)z}}{e^{\pi\mathbf{i}z} - e^{-\pi\mathbf{i}z}}$ as $\dfrac{\sin((2k+1)\pi z)}{\sin(\pi z)}$ we get the following equation, which was written in §35.1 above.

$$\boxed{\begin{gathered}\left(\prod_{n=1}^{\infty}\left(1 - e^{2\pi\mathbf{i}n\tau}\right)\right) \cdot \left(\prod_{n=1}^{\infty}\left(1 - e^{2\pi\mathbf{i}n\tau} e^{2\pi\mathbf{i}z}\right)\right) \cdot \left(\prod_{n=1}^{\infty}\left(1 - e^{2\pi\mathbf{i}n\tau} e^{-2\pi\mathbf{i}z}\right)\right) \\ = \sum_{k=0}^{\infty} (-1)^k e^{\pi\mathbf{i}\tau k(k+1)} \frac{\sin((2k+1)\pi z)}{\sin(\pi z)}\end{gathered}}$$

[5] a proof of this identity is given in Optional Reading B.