ALGEBRA 2. PROBLEM SET 1

Unless otherwise stated, the functors below are covariant.
References are made here to the lecture notes. For example, §2.5 refers to Lecture 2, paragraph 5^{th} (also labelled as (2.5) in Lecture 2).

Problem 1. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor from a category $\mathcal{C}$ to another category $\mathcal{D}$. Assume that F is faithful. For a morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, prove that if $F(f)$ is injective (resp. surjective) then so is f.

Problem 2. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Let f be a morphism in $\mathcal{C}$. Assume that f has a left (resp. right) inverse. Prove that the same is true for $F(f)$. Is the similar assertion true, where instead we have f injective (resp. surjective)?

Problem 3. Give an example for each of the following:

(1) A functor which is faithful and dense but not full.
(2) A functor which is full and dense, but not faithful.
(3) A functor which is faithful and full, but not dense.

Problem 4. Consider the following category, denoted in §0.7 by $\mathcal{F}(\mathbf{Ab})$. Its objects are *filetered abelian groups*: $G_\bullet = G_0 \supseteq G_1 \supseteq \ldots$ where G_j is an abelian group and G_{j+1} is a subgroup of G_j (for every $j \geq 0$). Morphisms are defined by:

$$\mathrm{Hom}(G_\bullet, H_\bullet) = \{f : G_0 \to H_0 \text{ group homomorphism such that } f(G_j) \subset H_j,\ \forall\ j \geq 0\}$$

Give an example of a morphism in $\mathcal{F}(\mathbf{Ab})$ which is a bijection but not an isomorphism.

Problem 5. Let M be a monoid and let $\underline{M}$ denote the category which has only one object, say p, and $\mathrm{End}_{\underline{M}}(p) = M$. Prove that functors between two such categories $\underline{M}$ and $\underline{N}$ are same as homomorphisms of monoids $F : M \to N$. Verify that natural transformations between two functor F, G correspond to elements $b \in N$ such that $b.F(x) = G(x).b$ for every $x \in M$.

Problem 6. Let G be a group and let G-**Sets** be the category whose objects are sets together with a G–action, and morphisms are set maps which commute with the G–action (see §2.7). Prove that natural isomorphisms of the forgetful functor $F : G\text{-}\mathbf{Sets} \to \mathbf{Sets}$ are given by elements of the group G. That is, $\mathrm{Aut}(F) = G$.

Problem 7. Let $\mathcal{C}$ be a category. Recall from §0.10 that for an object $X \in \mathcal{C}$, we have two (covariant and contravariant respectively) functors $h_X, h^X : \mathcal{C} \to \mathbf{Sets}$. Given a morphism $f : X \to Y$ in $\mathcal{C}$, prove that $h_f : h_Y \Rightarrow h_X$ and $h^f : h^X \Rightarrow h^Y$, defined in §1.5, are natural transformations.

Problem 8. Let G be a group and $H < G$ a subgroup. Verify that $\mathrm{Ind}_H^G : H\text{-}\mathbf{Sets} \to G\text{-}\mathbf{Sets}$ defined in §2.7 is a functor. Prove that $(\mathrm{Ind}_H^G, \mathrm{Res}_H^G)$ is an adjoint pair (see §2.7 for the definition of these functors).

Problem 9. Recall the definition of the category $\mathbf{Mat}_K$ from §1.7, where K is a fixed field. Recall that we have a functor $F : \mathbf{Mat}_K \to \mathbf{Vect}_K^{\mathrm{fd}}$.

(1) Prove that F is an equivalence of categories.
(2) Prove that constructing a functor $G : \mathbf{Vect}_K^{\mathrm{fd}} \to \mathbf{Mat}_K$ together with natural isomorphisms
$$\phi : \mathrm{Id}_{\mathbf{Mat}_K} \Rightarrow G \circ F \qquad \text{and} \qquad \psi : \mathrm{Id}_{\mathbf{Vect}_K^{\mathrm{fd}}} \Rightarrow F \circ G,$$
is same as making a choice of a basis $B_G(V)$ for every $V \in \mathbf{Vect}_K^{\mathrm{fd}}$.
(3) Let G_1, G_2 be two such functors, obtained by choosing $B_1(V), B_2(V)$ respectively, two bases of V, for every finite–dimensional vector space V. Prove that the change of basis matrix provides a natural isomorphism between G_1 and G_2.

Problem 10. Let R be a unital ring, and let $R\text{-}\operatorname{mod}$ be the category of left R–modules. Consider the forgetful functor $F : R\text{-}\operatorname{mod} \to \mathbf{Ab}$. What are the natural transformations $F \Rightarrow F$?

Problem 11. Let $\mathbf{Gps}$ be the category of groups and $\mathbf{Ab}$ the category of abelian groups. Consider the functor of natural inclusion $I : \mathbf{Ab} \to \mathbf{Gps}$. Let $A : \mathbf{Gps} \to \mathbf{Ab}$ be given, on objects by:
$$A : G \mapsto G/[G, G],$$
where $[G, G]$ is the subgroup of G generated by commutators $\{aba^{-1}b^{-1} : a, b \in G\}$ (recall that it is automatically normal). Prove that (A, I) is an adjoint pair.

Problem 12. Recall the definition of a pair of adjoint functors from §2.4. Let $F_1 : \mathcal{C} \to \mathcal{D}$ and $F_2 : \mathcal{D} \to \mathcal{C}$ be functors such that (F_1, F_2) is an adjoint pair.

(1) Prove that there is a natural transformation $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow F_2 \circ F_1$ (it is called *the unit of adjunction.*)
(2) Prove that there is a natural transformation $\varepsilon : F_1 \circ F_2 \Rightarrow \mathrm{Id}_{\mathcal{D}}$ (it is called *the counit of adjunction.*)