ALGEBRA 2. PROBLEM SET 2

If nothing is specified for a functor, it is assumed to be covariant.

Problem 1. Write the unit and counit of the following pairs of adjoint functors.

(1) Let G be a group and consider $\mathbf{Sets} \underset{F}{\overset{I}{\rightleftarrows}} G\text{-}\mathbf{Sets}$. Here, I sends a set X to $G \times X$, with G–action via left multiplication on the first component. F is the forgetful functor.

(2) $\mathbf{Sets} \underset{F_2}{\overset{F_1}{\rightleftarrows}} \mathbf{Gps}$. Here F_1 sends a set X to the free group generated by X, and F_2 is the forgetful functor.

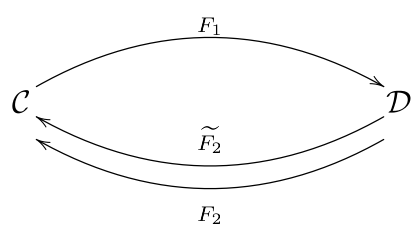
Problem 2. Assume that a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories (i.e, it is faithful, full and essentially surjective). Prove that F admits both a left, and a right adjoint.

Problem 3. Fix two sets $X, Y \in \mathbf{Sets}$ and define a functor $F : \mathbf{Sets} \to \mathbf{Sets}$ as follows: $F(Z) = \mathrm{Hom}_{\mathbf{Sets}}(X, Z) \times \mathrm{Hom}_{\mathbf{Sets}}(Y, Z)$ for every $Z \in \mathbf{Sets}$. For every morphism $f : Z \to Z'$ in $\mathbf{Sets}$, $F(f)$ is componentwise composition with f. That is, $F(f)\colon (g_1, g_2) \mapsto (f \circ g_1, f \circ g_2)$. Is this functor representable?

Problem 4. With the notations of problem 2 above, let G be the contravariant analogue of F. That is, $G(Z) = \mathrm{Hom}_{\mathbf{Sets}}(Z, X) \times \mathrm{Hom}_{\mathbf{Sets}}(Z, Y)$. Determine whether G is representable.

Problem 5. For a set X, let $\mathfrak{P}(X)$ be the set of all subsets of X. Consider the *covariant* functor $X \mapsto \mathfrak{P}(X)$ which sends $f : X \to Y$ to $\widetilde{f} : \mathfrak{P}(X) \to \mathfrak{P}(Y)$ given by: $\widetilde{f}(X') = f(X') \subset Y$, for every $X' \subset X$. Prove that this functor is **not** representable.

Problem 6. Consider the functors given in the figure below. Assume that (F_1, F_2) and $(F_1, \widetilde{F_2})$ are both adjoint pairs. Prove that there is a natural isomorphism (of functors) between F_2 and $\widetilde{F_2}$.



Problem 7. Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and $F : \mathcal{C} \to \mathcal{D}$ be a functor. Prove that F admits a right adjoint if, and only if, the following (contravariant) functor is representable, for every $Y \in \mathcal{D}$:

$$I_Y : \mathcal{C} \to \mathbf{Sets}, \qquad X \mapsto \mathrm{Hom}_{\mathcal{D}}(F(X), Y).$$

(You do not have to prove that I_Y is a functor, since it is a composition $I_Y = \mathrm{Hom}_{\mathcal{D}}(-, Y) \circ F$ of functors.)

Problem 8. Let F be a functor $\mathcal{C} \to \mathbf{Sets}$. Prove that F is represented by (X, ι_X) if, and only if there exists a distinguished element $p_X \in F(X)$ such that the following "universal property" holds.
For every $Y \in \mathcal{C}$ and $b \in F(Y)$, there is a unique morphism $f : X \to Y$ such that $F(f)(p_X) = b$.

Problem 9. Let K be a field, and let us fix two K–vector spaces V_1 and V_2. Consider the following functor:

$$T_{V_1,V_2} : \mathbf{Vect}_K \to \mathbf{Vect}_K$$

which sends W to the space of all K–bilinear maps $V_1 \times V_2 \to W$. State the universal property for a vector space V, so that it represents T_{V_1,V_2}. Give a construction of such a V.

Problem 10. Fix two groups G_1 and G_2 and consider the following functor

$$I : \mathbf{Gps} \to \mathbf{Sets}, \qquad H \mapsto \mathrm{Hom}_{\mathbf{Gps}}(G_1, H) \times \mathrm{Hom}_{\mathbf{Gps}}(G_2, H).$$

Determine whether this functor is representable.