ALGEBRA 2. HOMEWORK 3

$(I,\leq)$ is a partially ordered set, and $\mathbf{I}$ is the category associated to it, as in §6.2. Recall that, for a category $\mathcal{C}$, $\mathcal{F}(\mathbf{I},\mathcal{C})$ is the category of direct systems valued in $\mathcal{C}$ and $\mathcal{F}(\mathbf{I}^{\mathrm{op}},\mathcal{C})$ is the category of inverse systems valued in $\mathcal{C}$.

Problem 1. Assume that $(I,\leq)$ has a unique maximal element $i_0 \in I$ (that is, $i \leq i_0$ for every $i \in I$). Prove that for any direct system $\{(X_i),(\psi_{ji} : X_i \to X_j)\}$, we have an isomorphism $\varinjlim_{(I,\leq)} X_i \xrightarrow{\sim} X_{i_0}$.

Problem 2. With the same hypothesis as in Problem 1, prove that for any inverse system $\{(Y_i),(\varphi_{ij} : Y_j \to Y_i)\}$, we have an isomorphism $Y_{i_0} \xrightarrow{\sim} \varprojlim_{(I,\leq)} Y_i$.

Problem 3. Let $\mathcal{C}$ be a category such that for every $\mathfrak{X} \in \mathcal{F}(\mathbf{I},\mathcal{C})$, its direct limit exists.

(1) Prove that this gives a functor:
$$\varinjlim : \mathcal{F}(\mathbf{I},\mathcal{C}) \to \mathcal{C},$$
sending a direct system to its limit.

(2) Consider the constant functor $C : \mathcal{C} \to \mathcal{F}(\mathbf{I},\mathcal{C})$, which associates to each $X \in \mathcal{C}$, the direct system $C(X) = \{(X_i)_{i\in I},(\psi_{ji} : X_i \to X_j)_{i\leq j}\}$ with $X_i = X$ for every $i \in I$ and $\psi_{ji} = \mathrm{Id}_X$ for every $i \leq j$. Prove that $\left(\varinjlim, C\right)$ is an adjoint pair of functors.

Problem 4. Let $\mathcal{C}$ be a category such that for every $\mathfrak{Y} \in \mathcal{F}(\mathbf{I}^{\mathrm{op}},\mathcal{C})$, its inverse limit exists.

(1) Prove that this gives a functor:
$$\varprojlim : \mathcal{F}(\mathbf{I}^{\mathrm{op}},\mathcal{C}) \to \mathcal{C},$$
sending an inverse system to its limit.

(2) Again, consider the constant functor $C : \mathcal{C} \to \mathcal{F}(\mathbf{I}^{\mathrm{op}},\mathcal{C})$, which associates to each $Y \in \mathcal{C}$, the inverse system $C(Y) = \{(Y_i)_{i\in I},(\varphi_{ij} : Y_j \to Y_i)_{i\leq j}\}$ with $Y_i = Y$ for every $i \in I$ and $\varphi_{ij} = \mathrm{Id}_Y$ for every $i \leq j$. Prove that $\left(C, \varprojlim\right)$ is an adjoint pair of functors.

Problem 5. (Same hypothesis on $\mathcal{C}$ as in Problem 3 above). Assume that we have two direct systems $\mathfrak{X}_1, \mathfrak{X}_2 \in \mathcal{F}(\mathbf{I},\mathcal{C})$:
$$\mathfrak{X}_\ell = \left\{ \left(X_i^{(\ell)}\right)_{i\in I}, \left(\psi_{ji}^{(\ell)} : X_i^{(\ell)} \to X_j^{(\ell)}\right)_{i\leq j} \right\}, \qquad \ell = 1,2.$$
Let $(f_i)_{i\in I}$ be a morphism in $\mathcal{F}(\mathbf{I},\mathcal{C})$ from $\mathfrak{X}_1$ to $\mathfrak{X}_2$, such that f_i is surjective, for every $i \in I$. Prove that $\varinjlim f_i : \varinjlim X_i^{(1)} \to \varinjlim X_i^{(2)}$ is surjective.

Problem 6. (Same hypothesis on $\mathcal{C}$ as in Problem 4 above). Prove that inverse limit of injective morphisms is injective (see Problem 5, where this statement is written in detail for direct limits).

Problem 7. Assume $(I, \leq)$ is right–directed (i.e, for every $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$). Give a construction of the direct limit over $(I, \leq)$ in the category of groups **Gps**.

Problem 8. Let $G \in \mathbf{Ab}$ be an abelian group. Consider the direct system in **Ab**, whose elements are *finitely generated* subgroups of G, and morphisms are natural inclusions. Thus, the partially ordered set labels finitely generated subgroups of G: $\{G_i\}_{i \in I}$, and $i \leq j$ means $G_i \subset G_j$. Verify that this is a right–directed poset, and prove that $\varinjlim_{(I,\leq)} G_i \xrightarrow{\sim} G$.

Problem 9. Let $(I, \leq) = \mathbb{N}$ with its usual total order. Consider the following two inverse systems, valued in **Ab**:

(InvSys1) $X_n = \mathbb{Z}$ for every $n \in \mathbb{N}$, and for every $m < n$, $\varphi_{mn} : \mathbb{Z} \to \mathbb{Z}$ is multiplication by 3^{n-m}.

(InvSys2) $X_n = \mathbb{Z}/2\mathbb{Z}$ for every $n \in \mathbb{N}$, and all the morphisms are identities.

1. Prove that the inverse limit of (InvSys1) is $\{0\}$.
2. Prove that the canonical surjection $\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ is a morphism from (InvSys1) to (InvSys2), whose inverse limit is $\{0\} \to \mathbb{Z}/2\mathbb{Z}$.

(*Hence, inverse limit of surjective maps need not be surjective.*)

Problem 10. Again take $(I, \leq) = \mathbb{N}$. For each $n \in \mathbb{N}$, let $A_n = \mathbb{C}[x]/(x^{n+1})$ and for each $m < n$, $\pi_{mn} : A_n \to A_m$ the natural projection (see §7.3).

1. Prove that in the category of commutative rings, $\mathbb{C}[\![x]\!] \xrightarrow{\sim} \varprojlim_{n \in \mathbb{N}} A_n$.
2. Consider the category of $\mathbb{N}$–*graded* commutative rings. An object of this category is a (unital) commutative ring R, together with abelian subgroups $R[k] (k \in \mathbb{N})$ such that
   - $R = \bigoplus_{k=0}^{\infty} R[k]$ as an abelian group.
   - $x \in R[k]$ and $y \in R[\ell]$ implies $xy \in R[k + \ell]$.

   Prove that, in this category $\mathbb{C}[x] \xrightarrow{\sim} \varprojlim_{n \in \mathbb{N}} A_n$.