ALGEBRA 2. HOMEWORK 4

Problem 1. Let $\mathcal{A}$ be an abelian category.

(1) Show that a sequence of morphisms $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is exact if, and only if for every $X \in \mathcal{A}$, the following is an exact sequence in **Ab**:

$$0 \to \mathrm{Hom}_{\mathcal{A}}(X, A) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathcal{A}}(X, B) \xrightarrow{g\circ -} \mathrm{Hom}_{\mathcal{A}}(X, C).$$

(2) Show that a sequence of morphisms $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact if, and only if for every $X \in \mathcal{A}$, the following is an exact sequence in **Ab**:

$$0 \to \mathrm{Hom}_{\mathcal{A}}(C, X) \xrightarrow{-\circ g} \mathrm{Hom}_{\mathcal{A}}(B, X) \xrightarrow{-\circ f} \mathrm{Hom}_{\mathcal{A}}(A, X).$$

Problem 2. Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. Consider a pair of additive covariant functors $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$. Assume that (F, G) is an adjoint pair. Prove that F is right exact, and G is left exact.

Problem 3. Assume that $\mathcal{C}$ is an additive category in which arbitrary direct sums and products exist. Further assume that direct sums and products are isomorphic. Prove that $\mathcal{C}$ is trivial (that is, every object of $\mathcal{C}$ is isomorphic to the trivial one).

In problems 4–6 below, $\mathcal{A} = R$-mod is the category of left R–modules for a unital ring R.

Problem 4. Let J be a set. Prove that $\bigoplus_J, \prod_J : \mathcal{A}^J \to \mathcal{A}$ are exact functors.

Problem 5. Let $(I, \leq)$ be a *right directed* partially ordered set. Prove that the direct limit $\varinjlim_{(I,\leq)} : \mathcal{F}(\mathbf{I}, \mathcal{A}) \to \mathcal{A}$ is an exact functor. *Recall that* $\mathbf{I}$ *is the category associated to the partially ordered set* $(I, \leq)$ *and* $\mathcal{F}(\mathbf{I}, \mathcal{A})$ *is the category of direct systems valued in* $\mathcal{A}$.

Problem 6. Let $(I, \leq)$ be a partially ordered set. Prove that the inverse limit $\varprojlim_{(I,\leq)} : \mathcal{F}(\mathbf{I}^{\mathrm{op}}, \mathcal{A}) \to \mathcal{A}$ is left exact, but not right exact.

In problems 7–10, $\mathcal{A} = A$-mod for a unital commutative ring A.

Problem 7. Let I be a set, $\{M_i\}_{i\in I}$ a set of A–modules. For $N \in A$-mod, show that we have an isomorphism:

$$\left(\bigoplus_{i\in I} M_i\right) \otimes N \cong \bigoplus_{i\in I} (M_i \otimes N).$$

Problem 8. Prove or give a counterexample to the statement of the previous problem, if direct sum (on both sides of the equation) is replaced by direct product.

Problem 9. Let $(I, \leq)$ be a *right directed* poset, and $\{(M_i)_{i\in I}, (\psi_{ji} : M_i \to M_j)_{i\leq j}\}$ a directed system of A–modules. Prove that, for every $N \in A$-mod:

$$\left(\varinjlim_{i\in I} M_i\right) \otimes N \cong \varinjlim_{i\in I}(M_i \otimes N).$$

Problem 10. Is the statement from the previous problem true for inverse limits?