ALGEBRA 2. HOMEWORK 5

In the problems below, $\mathcal{A} = R$-mod is the abelian category of left R–modules, over a unital ring R. $\mathbb{K}^\bullet(\mathcal{A})$ is the category of complexes of R–modules.

Problem 1.– Consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
M_1 & \longrightarrow & M_2 & \xrightarrow{a} & M_3 & \longrightarrow & M_4 \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle s} \\
N_1 & \longrightarrow & N_2 & \xrightarrow{b} & N_3 & \longrightarrow & N_4
\end{array}$$

Assume that u is surjective and s is injective. Prove that $\mathrm{Ker}(w) = a(\mathrm{Ker}(v))$ and $\mathrm{Im}(v) = b^{-1}(\mathrm{Im}(w))$.

Problem 2.– Assume we have two morphisms of R–modules $M \xrightarrow{u} N \xrightarrow{v} P$. Prove that we get the following *exact sequence*:

$$0 \to \mathrm{Ker}(u) \to \mathrm{Ker}(v \circ u) \to \mathrm{Ker}(v) \to \mathrm{CoKer}(u) \to \mathrm{CoKer}(v \circ u) \to \mathrm{CoKer}(v) \to 0.$$

Problem 3.– (Pull–backs). Given two morphisms of R–modules

$$\begin{array}{ccc}
 & & B \\
 & & \downarrow{\scriptstyle g} \\
A & \xrightarrow{f} & S
\end{array}$$

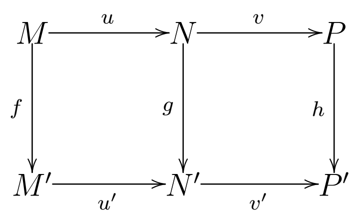
$A \times_S B$ is defined as $\{(a, b) : f(a) = g(b)\} \subset A \times B$, with canonical maps $p_1 : A \times_S B \to A$ and $p_2 : A \times_S B \to B$, given by $p_1(a, b) = a$ and $p_2(a, b) = b$ (see Problem 3 of Mid Term 1, for a more categorical definition).

Let $M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of R–modules, and assume that we have a morphism $X \to M_3$. Define $\widetilde{X_2} = M_2 \times_{M_3} X$ and $\widetilde{X_1} = M_1 \times_{M_2} \widetilde{X_2}$. Prove that the first row of the following diagram is exact:

$$\begin{array}{ccccccc}
\widetilde{X_1} & \longrightarrow & \widetilde{X_2} & \longrightarrow & X & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0
\end{array}$$

Problem 4.– State and prove the result analogous to that of Problem 3, for "push–forwards".

Problem 5.– Consider the following diagram in $\mathcal{A}$ with exact rows and commuting squares.



(1) Prove that, if u', f, h are injective, then g is injective.

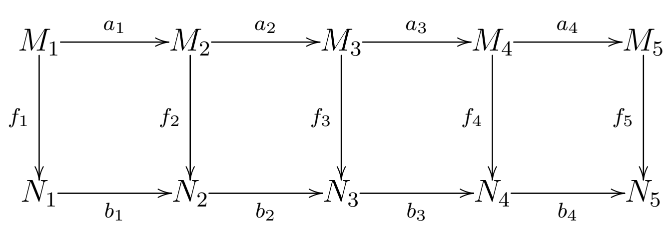
(2) Prove that, if v, f, h are surjective, then g is surjective.

(3) Prove that, if g is injective and f, v are surjective, then h is injective.

(4) Prove that, if g is surjective and h, u' are injective, then f is surjective.

Problem 6.– Consider the subcategory of $\mathbb{K}^\bullet(\mathcal{A})$ consisting of exact complexes (i.e, $C^\bullet$ such that $H^n(C^\bullet) = 0$ for every $n \in \mathbb{Z}$). Give an example to illustrate that this subcategory is not closed under taking kernels.

Problem 7.– *Five lemma.* Consider the following commutative diagram with exact rows.

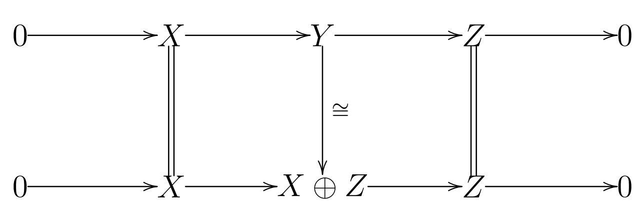


(1) Prove that if f_2, f_4 are injective and f_1 is surjective, then f_3 is injective.

(2) Prove that if f_2, f_4 are surjective and f_5 is injective, then f_3 is surjective.

Hence, if f_1, f_2, f_4, f_5 are isomorphisms, then so is f_3.

Problem 8.– Let $\mathcal{C}$ be an arbitrary abelian categories. A short exact sequence in $\mathcal{C}$, $0 \to X \to Y \to Z \to 0$ is said to be *split* if there exists an isomorphism $Y \xrightarrow{\sim} X \oplus Z$ making the following diagram commute:



Prove that a sequence $0 \to X \xrightarrow{i} Y \xrightarrow{\pi} Z \to 0$ is split if, and only if there exists $s : Z \to Y$ such that $\pi \circ s = \mathrm{Id}_Z$.

Problem 9.– Let $C^\bullet$ and $D^\bullet$ be two cochain complexes of R–modules. Let $\alpha^\bullet : C^\bullet \to D^\bullet$ be a morphism. Define:

- $C[1]^\bullet \in \mathbb{K}^\bullet(\mathcal{A})$ is a complex with $C[1]^n = C^{n+1}$ and $d^n_{C[1]} = -d^{n+1}_C$, for every $n \in \mathbb{Z}$.
- $\mathrm{Cone}(\alpha)$ is the following complex:
  - For every $n \in \mathbb{Z}$, $\mathrm{Cone}(\alpha)^n = C^{n+1} \oplus D^n$.
  - For every $n \in \mathbb{Z}$, the differential, denoted by d^n below, is given by the following expression, for every $x \in C^{n+1}$ and $y \in D^n$,
  $$\mathsf{d}^n(x, y) = (-d^{n+1}_C(x), d^n_D(y) - \alpha^{n+1}(x))$$

(1) Verify that $\mathrm{Cone}(\alpha)$ is a complex.

(2) Prove that we have the following short exact sequence of cochain complexes.
$$0 \to D^\bullet \to \mathrm{Cone}(\alpha) \to C^\bullet[1] \to 0$$

(3) Prove that $\alpha^\bullet$ is null homotopic if, and only if, the sequence above is split (see the previous problem for its definition).

(4) Prove that $H^n(\alpha)$ is an isomorphism, for every $n \in \mathbb{Z}$ if, and only if $H^n(\mathrm{Cone}(\alpha)) = 0$ for every $n \in \mathbb{Z}$.

Problem 10.– *Koszul complex.* Let K be a field, $A = K[x, y, z]$ be the (commutative) ring of polynomials in three variables, with coefficients from K. Let $A^{\oplus 3}$ be the A–module which is direct sum of three copies of A, with A–basis denoted by e_1, e_2, e_3. Consider the following sequence of morphisms:
$$0 \to A \xrightarrow{d_2} A^{\oplus 3} \xrightarrow{d_1} A^{\oplus 3} \xrightarrow{d_0} A \to 0$$

- $d_0(e_1) = x$, $d_0(e_2) = y$, and $d_0(e_3) = z$.
- $d_1(e_1) = ze_2 - ye_3$, $d_1(e_2) = xe_3 - ze_1$, and $d_1(e_3) = ye_1 - xe_2$.
- $d_2(1) = xe_1 + ye_2 + ze_3$.

Verify that this is a complex of A–modules. Prove that it is exact at each spot, except $\mathrm{CoKer}(d_0) = K \cong K[x, y, z]/(x, y, z)$.

Bonus. Generalize this to the polynomial ring of n–variables.