ALGEBRA 2. HOMEWORK 6

R-mod is the abelian category of left R–modules, over a unital ring R.

Problem 1.– Let I be a set and let $R^{(I)}$ be the direct sum of I copies of R, viewed as a left R–module.

$$R^{(I)} = \bigoplus_{i\in I} M_i, \qquad \text{where } M_i = R,\ \forall i \in I.$$

Prove that $R^{(I)}$ is a projective module. *Any module isomorphic to $R^{(I)}$ for some indexing set I is called a free R–module. This problem is asking you to prove that free implies projective.*

Problem 2.– Prove that an R–module P is projective if, and only if there exists P' such that $P \oplus P'$ is free (see problem 1).

Problem 3.– Prove that R-mod has enough projectives, using Problem 1.

Problem 4.– Assume that there exists $p \in R$ such that $p^2 = p$. Prove that $M = Rp$ (left ideal generated by p) is a projective R–module.

Problem 5.– Prove that $\mathbb{Z}/3\mathbb{Z}$, is a projective $\mathbb{Z}/6\mathbb{Z}$–module.

Problem 6.– Let $M \in R$-mod and assume that we have two short exact sequences with P_1, P_2 projectives:

$$0 \to N_1 \to P_1 \to M \to 0, \qquad \text{and} \qquad 0 \to N_2 \to P_2 \to M \to 0.$$

Prove that $P_1 \oplus N_2 \cong P_2 \oplus N_1$.

Problem 7.– Prove that direct sum of projective modules is projective. Prove that direct product of injective modules is injective.

Problem 8.– Assume that R is commutative, and P is a projective R–module. Prove that $P \otimes - : R\text{-mod} \to R\text{-mod}$ is exact.

Problem 9.– Let $N \in \mathbb{Z}_{\geq 2}$. Verify that $0 \to \mathbb{Z} \xrightarrow{\mu_N} \mathbb{Z} \to 0$, where μ_N is multiplication by N, is a projective resolution of $\mathbb{Z}/N\mathbb{Z} \in \mathbb{Z}$-mod.

Problem 10.– Prove that the following infinite complex is a projective resolution of $\mathbb{Z}/3\mathbb{Z}$ as a module over $\mathbb{Z}/9\mathbb{Z}$:

$$\cdots \xrightarrow{\mu_3} \mathbb{Z}/9\mathbb{Z} \xrightarrow{\mu_3} \mathbb{Z}/9\mathbb{Z} \xrightarrow{\mu_3} \mathbb{Z}/9\mathbb{Z} \to 0.$$

Problem 11.– Assume that R is commutative, and $M, N \in R$-mod. Let $Q \in \mathbf{Ab} = \mathbb{Z}$-mod. Prove that we have an isomorphism of R–modules:

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(N, Q)) \cong \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R N, Q),$$

where, for an R–module X and an abelian group Y, the abelian group $\mathrm{Hom}_{\mathbb{Z}}(X, Y)$ has a structure of an R–module via:

$$r \in R,\ f \in \mathrm{Hom}_{\mathbb{Z}}(X, Y) \qquad \rightsquigarrow \qquad (r \cdot f)(x) = f(rx).$$