ALGEBRA 2. HOMEWORK 7

R denotes a unital commutative ring. R-mod is the category of R–modules.

Problem 1.– Let $P \in R$-mod. Prove that the following are equivalent:

(1) P is projective.
(2) $\mathrm{Ext}^k(P, N) = 0$ for every $k \in \mathbb{Z}_{\geq 1}, N \in R$-mod.
(3) $\mathrm{Ext}^1(P, N) = 0$ for every $N \in R$-mod.

Problem 2.– Let R be a principal ideal domain, and $0 \neq a \in R$. Prove that $R/(a)$ is an injective $R/(a)$–module.

Problem 3.– Let $\mathbb{Q}(x)$ be the field of rational functions in one variable, with coefficients from $\mathbb{Q}$. Prove that $\mathbb{Q}(x)/\mathbb{Z}[x]$ is a divisible $\mathbb{Z}[x]$–module, and it is not injective.

Problem 4.– Assume we have $a, b \in R$ such that (i) a is not a zero divisor and (ii) b is not a zero divisor in $R/(a)$. Prove that the following is a free resolution of $R/(a, b)$. Here $R^2 = R \oplus R$, and the maps are written in the usual matrix notation.

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} b \\ -a \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} a & b \end{bmatrix}} R \longrightarrow 0$$

Problem 5.– Assume $a, b \in R$ are such that $(0 : a) := \{r \in R : ra = 0\} = (b)$, and $(0 : b) = (a)$. Prove that the following infinite chain complex is a free resolution of $R/(a)$. Here μ_x is the multiplication by x map.

$$\cdots \xrightarrow{\mu_b} R \xrightarrow{\mu_a} R \xrightarrow{\mu_b} R \xrightarrow{\mu_a} R \to 0.$$

Problem 6.– Let K be a field and consider a finite complex $V^\bullet$ of K–vector spaces:

$$0 \to V^0 \to V^1 \to \cdots \to V^n \to 0.$$

Prove that $\displaystyle\sum_{i=0}^{n} (-1)^i \dim(H^i(V^\bullet)) = 0$ (rank–nullity theorem).

Problem 7.– Assume that R is an integral domain. Prove that every flat R–module is torsion free.

Problem 8.– Assume that $M, N \in R$-mod are two flat modules. Prove that $M \otimes N$ is again flat.

Problem 9.– Compute $\mathrm{Ext}^k_R(M, N)$ in each of the following case.

(1) $R = \mathbb{Z}[x]$ and $M = N = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}[x]/(2, x)$.
(2) $R = K[x, y]$ (here K is a field) and $M = N = K$.
(3) $R = \mathbb{Z}/4\mathbb{Z}$ and $M = N = \mathbb{Z}/2\mathbb{Z}$.

Problem 10.– Compute $\mathrm{Tor}_k^R(M, N)$ in the problems below. Here $R = K[x, y]$, where K is a field.

(1) $M = K[x, y]/(y)$ and $N = K[x, y]/(y - x^2)$.
(2) $M = N = K[x] = K[x, y]/(y)$.
(3) $M = N = K = K[x, y]/(x, y)$.

Problem 11.– Assume that we have a short exact sequence of R–modules, where F is a flat R–module: $0 \to N \to M \to F \to 0$. Prove that, for every R–module E, the following sequence is exact: $0 \to N \otimes E \to M \otimes E \to F \otimes E \to 0$.

Hint for problem 11: take a projective module P, with a surjective map $P \to E$. Let K be the kernel of this surjection. Draw a snake lemma–style diagram where you tensor the given short exact sequence with K for the top row and P for the bottom row.

Problem 12.– Again, assume that there is a short exact sequence of R–modules, where F is a flat R–module: $0 \to F_1 \to F_2 \to F \to 0$. Prove that F_1 is flat if, and only if F_2 is flat.

Hint for problem 12: any given injective morphism $A \to B$ will give rise to a snake–lemma–type picture, when tensored with the given short exact sequence, just like in problem 6 above.

The following problems are included to help you refresh your memory about rings and modules of fractions. Here R is again a unital commutative ring, and $S \subset R$ is a multiplicatively closed set (meaning: $0 \notin S$, $1 \in S$ and $a, b \in S \Rightarrow ab \in S$).

Problem R1.– $S^{-1}R$ represents the following covariant functor. Let **CommRings** be the category of commutative (unital) rings. F : **CommRings** $\to$ **Sets** is defined on objects as:

$$A \mapsto \{f : R \to A \text{ ring homomorphism, such that } f(S) \subset A^\times\}.$$

Here $A^\times$ is the (multiplicative) group of invertible elements of A.

Problem R2.– $S^{-1}R$ is a free R–module if, and only if $S \subset R^\times$ (in other words, $S^{-1}R = R$).

Problem R3.– For any $M \in R$-mod, $S^{-1}R \otimes_R M = S^{-1}M$.

Problem R4.– For every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$, the following sequence is exact:

$$0 \to S^{-1}M_1 \to S^{-1}M_2 \to S^{-1}M_3 \to 0.$$

Problem R5.– The kernel of the natural R–linear map $N \to S^{-1}N$ is $\{n \in N : \exists s \in S \text{ such that } sn = 0\}$.