ALGEBRA 2. HOMEWORK 9

Problem 1.– Let K be a field and let $p(x) \in K[x]$ be a monic polynomial of degree n polynomial. Let L/K be the splitting extension of $p(x)$. Prove that $[L : K]$ divides $n!$.

Problem 2.– Prove that $f(x) = x^6+x^3+1 \in \mathbb{Q}[x]$ is irreducible. Determine its 6 roots in $\mathbb{C}$.

Problem 3.– Prove that the splitting extension of $x^3 - 2$ over $\mathbb{Q}$ has degree 6.

Problem 4.– Let L/K be an algebraic extension. Let $\sigma : L \to L$ be a ring homomorphism such that $\sigma|_K = \mathrm{Id}_K$. Prove that σ is then an isomorphism.

Problem 5.– Give an example to prove that the assertion of Problem 1 is false, if E/F is not assumed to be algebraic.

Problem 6.– Let $\zeta = \exp\left(\frac{2\pi\iota}{7}\right) \in \mathbb{C}$. Prove that $\mathbb{Q}(\zeta)$ is a Galois extension of $\mathbb{Q}$.

Problem 7.– Let $k \subset K$ be two fields, and assume that K is algebraically closed. Let K' consist of all elements of K which are algebraic over k. Prove that K' is isomorphic to the algebraic closure of k.

Problem 8.– Let $a, b \in \mathbb{Q}$ and consider $f(x) = x^3 - ax + b \in \mathbb{Q}[x]$. Let L be the splitting extension of $f(x)$ over $\mathbb{Q}$. If $\alpha, \beta, \gamma \in L$ are the roots of f, then set $\Delta = (\alpha-\beta)^2(\alpha-\gamma)^2(\beta-\gamma)^2 \in \mathbb{Q}$ ((-1) times the discriminant of f, as defined in Lecture 27). Assuming $f(x)$ is irreducible, prove that $[L : \mathbb{Q}] = 6$ if and only if $\sqrt{\Delta} \notin \mathbb{Q}$.

Problem 9.– Let K be a field and let $f(x), g(x) \in K[x]$ be two monic polynomials of degrees m, n respectively. Let L/K be the splitting extension of $\{f(x), g(x)\}$. Let $r_1, \dots, r_m; s_1, \dots, s_n \in L$ be the roots of $f(x)$ and $g(x)$ respectively.

(1) Prove that the following is an element of K:

$$\mathrm{Res}(f, g) = \prod_{\substack{1 \le i \le m \\ 1 \le j \le n}} (r_i - s_j),$$

called the resultant of f and g.

(2) Prove that there is a unique polynomial $P(A_1, \dots, A_m; B_1, \dots, B_n)$ in $m + n$ variables such that, for every $f(x) = x^m + \sum_{j=1}^m a_j x^{m-j}$ and $g(x) = x^n + \sum_{k=1}^n b_k x^{n-k}$, $\mathrm{Res}(f, g) = P(a_1, \dots, a_m; b_1, \dots, b_n)$.

(3) Prove that $\mathrm{Res}(f, g) = 0$ if and only if f and g have a common root in $\overline{K}$ (algebraic closure of K).

(4) For $f = x^2 + b_1x + c_1$ and $g = x^2 + b_2x + c_2$, compute $\mathrm{Res}(f, g)$.

Problem 10.– Let K be a field and let $L = K(T)$ be the field of rational functions in one variable T.

(1) Prove that for every $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(K)$, we have an element $\sigma_g \in \mathsf{G}(L/K)$ given by:
$$\sigma_g(T) = \frac{aT + b}{cT + d}.$$

(2) Let $\psi : \mathrm{GL}_2(K) \to \mathsf{G}(L/K)$ be given by $\psi(g) = \sigma_g$. Verify that ψ is a group homomorphism and $\mathrm{Ker}(\psi) = K^\times \,\mathrm{Id}_2$, where $K^\times = K \setminus \{0\}$ and Id_2 is the 2×2 identity matrix.

(3) Let $\mathrm{PGL}_2(K) = \mathrm{GL}_2(K)/\mathrm{Ker}(\psi)$. Prove that $\mathrm{PGL}_2(K) \cong \mathsf{G}(L/K)$.

(4) Assume that K is an infinite field. Prove that L/K is a Galois extension.

(Hint for (3). Let $y = \dfrac{g(T)}{h(T)} \in K(T)$, where g and h have no common factor. Define $\mathrm{ht}(y) := \max(\deg(g), \deg(h))$, height of y. Prove that, if $y \notin K$, then $g(X) - h(X)y \in (K(y))[X]$ is irreducible. Use this to show that, if $\mathrm{ht}(y) = n > 0$, then L is an algebraic extension of $K(y)$ of degree n.)