ALGEBRA 2. HOMEWORK 10

Problem 1.– Prove that every algebraic extension of a perfect field is perfect.

Problem 2.– Let $p \in \mathbb{Z}_{\geq 2}$ be a prime number and $\mathbb{F}_p$ be the field with p elements. Let λ be a variable and let $K = \mathbb{F}_p(\lambda)$ be the field of rational functions in λ with coefficients from $\mathbb{F}_p$.

For each $n \in \mathbb{Z}_{\geq 1}$, let K_n be the splitting extension of $x^{p^n} - \lambda \in K[x]$.

(1) Prove that we have a chain of field extensions:

$$K \subset K_1 \subset K_2 \subset \dots$$

(2) Prove that $\mathsf{G}(K_n/K) = \{\mathrm{Id}\}$ for every n.

(3) Let $\widetilde{K} = \bigcup_{n \geq 1} K_n$. Prove that $\widetilde{K}$ is a perfect field.

Problem 3.– Again, let $p \in \mathbb{Z}_{\geq 2}$ be a prime number. Consider $K = \mathbb{F}_p(\lambda, \mu)$ and L be the splitting extension of $\{x^p - \lambda, x^p - \mu\} \subset K[x]$. Prove or disprove: there exists $\gamma \in L$ such that $L \cong K(\gamma)$.

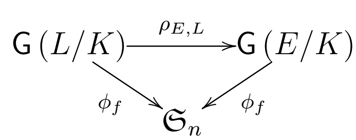
Problem 4.– Let L/K be a Galois extension. Let $\alpha \in L$ and $f(x) = \mathsf{m}_\alpha(x) \in K[x]$. Let $n = \deg(f) \in \mathbb{Z}_{\geq 1}$. Let $\{r_1, \dots, r_n\} \subset L$ be the set of n distinct roots of $f(x)$.

(1) Prove that we have a group homomorphism

$$\phi_f : \mathsf{G}(L/K) \to \mathfrak{S}_n,$$

given by $\phi_f(g) = \sigma$ if $g(r_i) = r_{\sigma(i)}$ for every $1 \leq i \leq n$.

(2) Let $E = K(r_1, \dots, r_n) \subset L$. Prove that the restriction homomorphism $\rho_{E,L}$: $\mathsf{G}(L/K) \to \mathsf{G}(E/K)$ commutes with ϕ_f. That is, the following diagram commutes:



(3) Prove that the action of $\mathsf{G}(E/K)$ on the set $\{1, \dots, n\}$, via ϕ_f, is transitive. That is, for any i, j there exits $g \in \mathsf{G}(E/K)$ such that $\phi_f(g) : i \mapsto j$.

Problem 5.– Let L/K be a normal extension. Let E be an intermediate field: $K \subset E \subset L$ and let $\sigma : E \to L$ be an embedding which is the identity on K. Prove that σ extends to an automorphism of L.

Problem 6.– Let $K = \mathbb{Q}$, E be the splitting field of $X^2 - 2 \in \mathbb{Q}[X]$ over $\mathbb{Q}$. Thus $E = \mathbb{Q}(\sqrt{2})$. Let L be the splitting field of $X^2 - \sqrt{2} \in E[X]$ over E. Describe the group $\mathsf{G}(L/K)$. Prove that L/K is not a Galois extension. *Splitting extension of a splitting extension is not necessarily splitting extension.*

Problem 7.– Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $K = \mathbb{Q}$. Compute $\mathsf{G}\,(L/K)$. Is L/K a Galois extension?

Problem 8.– Let $\alpha = \sqrt{2+\sqrt{2}} \in \mathbb{R}$. Compute $\mathsf{G}\,(\mathbb{Q}(\alpha)/\mathbb{Q})$.

Problem 9.– Let $n \in \mathbb{Z}_{\geq 2}$ and consider $L = \mathbb{Q}(T_1, \ldots, T_n)$. Let G be a subgroup of $\mathfrak{S}_n$, and let $K = L^G$. Prove that L/K is a Galois extension with $\mathsf{G}\,(L/K) = G$.

Assume that L/K is a Galois extension and let $\Gamma = \mathsf{G}\,(L/K)$. Given an L–vector space V, a K–*structure* on V is the data of a vector sub-K-subspace $V_0 \subset V$ such that the canonical map $L \otimes_K V_0 \to V$, given by $\alpha \otimes v \mapsto \alpha \cdot v$, is an isomorphism.

Let $V_0 \subset V$ be a K–structure on an L–vector space V. We get an action of Γ on V as follows. Let $\{e_i\}_{i\in I}$ be a basis of V_0. For any $v \in V$, write $v = \sum_{i\in I} \alpha_i e_i$, where $\alpha_i \in L$. Then, for any $\sigma \in \Gamma$ we define:

$$u_\sigma(v) = \sum_{i\in I} \sigma(\alpha_i) e_i.$$

Problem 10.– Let L/K, $V_0 \subset V$, Γ and $\{u_\sigma : V \to V\}_{\sigma\in\Gamma}$ be as above.

(1) Prove that $\sigma \mapsto u_\sigma$ is a group homomorphism $\Gamma \to \mathrm{Aut}_{K\text{–}vs}(V)$ such that for every $\alpha \in L$, $\sigma \in \Gamma$ and $v \in V$, we have:
$$u_\sigma(\alpha v) = \sigma(\alpha) u_\sigma(v).$$
(2) Let $x \in V$. Prove that $x \in V_0$ if and only if $u_\sigma(x) = x$ for every $\sigma \in \Gamma$.
(3) Let $U \subset V$ be a vector sub-L-subspace. Prove that $U_0 = U \cap V_0$ is a K–structure on U if and only if $u_\sigma(U) \subset U$ for every $\sigma \in \Gamma$. (In this case, we say U is rational over K).
(4) Let V, W be two L–vector spaces, together with K–structures $V_0 \subset V$ and $W_0 \subset W$. Let $f : V \to W$ be an L–linear map. Prove that $f(V_0) \subset W_0$ if and only if $f(u_\sigma(x)) = u_\sigma(f(x))$ for every $x \in V$. (In this case, we say f is rational over K).

Problem 11.– Let V be an L–vector space and assume that we are given a group homomorphism $u : \Gamma \to \mathrm{Aut}_{K\text{–}vs}(V)$ (denoted as before by $\sigma \mapsto u_\sigma$) such that

$$u_\sigma(\alpha v) = \sigma(\alpha) u_\sigma(v), \qquad \forall\, \sigma \in \Gamma,\ \alpha \in L, v \in V.$$

Let $V_0 = \{v \in V : u_\sigma(v) = v,\ \forall\, \sigma \in \Gamma\}$. Prove that if $|\Gamma| < \infty$ then $V_0 \subset V$ is a K–structure on V.

Problem 12.– (from Algebra 1). Let $A = \mathbb{Z}$, or $K[x]$. Let M be a finitely generated, torsion module over A. Prove that there exists $x \in M$ such that $\mathrm{Ann}(x) = \mathrm{Ann}(M)$. Here,

$$\mathrm{Ann}(x) = \{a \in A : ax = 0\}, \qquad \mathrm{Ann}(M) = \{a \in A : am = 0,\ \forall\, m \in M\}.$$