ALGEBRA 2. HOMEWORK 11

Problem 1.– Consider the following subgroups of $\mathbb{C}^\times$.

$$\mu_n(\mathbb{C}) := \{z \in \mathbb{C} : z^n = 1\}, \qquad \mu_\infty(\mathbb{C}) := \{z \in \mathbb{C} : z^r = 1 \text{ for some } r \in \mathbb{Z}_{\geq 1}\}.$$

Let $\mathrm{Z} = \mathbb{Z}_{\geq 1}$ with partial order given by divisibility.

(1) Show that $\{\mu_n(\mathbb{C})\}_{n\in\mathrm{Z}}$ is a direct system, and $\mu_\infty(\mathbb{C}) = \varinjlim_{n\in\mathrm{Z}} \mu_n(\mathbb{C})$.

(2) Show that $\mu_\infty(\mathbb{C}) \cong \mathbb{Q}/\mathbb{Z}$.

(3) Show that $\mathbb{Q}(\mu_\infty) = \varinjlim_{n\in\mathrm{Z}} \mathbb{Q}(\mu_n)$.

(4) Using $\mathsf{G}(\mathbb{Q}(\mu_n)/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$, show that $\mathsf{G}(\mathbb{Q}(\mu_\infty)/\mathbb{Q}) \cong \varprojlim_{n\in\mathrm{Z}} (\mathbb{Z}/n\mathbb{Z})^\times$. Describe a fundamental system of open neighbourhoods of $\mathrm{Id} \in \mathsf{G}(\mathbb{Q}(\mu_\infty)/\mathbb{Q})$.

Problem 2.– Recall the definition of cyclotomic polynomials. Let $P_n \subset \overline{\mathbb{Q}}$ be the set of primitive n^{th} roots of unity. Then $\{\Phi_n(x)\}_{n\geq 1}$ is defined as:

$$\Phi_n(x) = \prod_{a\in P_n} x - a$$

Note that $\Phi_1(x) = x - 1$.

(1) Show that $x^n - 1 = \prod_{d|n} \Phi_d(x)$. Use this to show that $\Phi_d(x) \in \mathbb{Z}[x]$.

(2) Let $p \geq 2$ be a prime number. Compute $\Phi_p(x)$. Show that $\Phi_{p^r}(x) = \Phi_p(x^{p^r})$.

(3) Let $\mu : \mathbb{Z}_{\geq 1} \to \{0, \pm 1\}$ be defined by:

$$\mu(p_1^{k_1} \dots p_h^{k_h}) = \begin{cases} 0 & \text{if } k_j \geq 2 \text{ for some } 1 \leq j \leq h \\ (-1)^h & \text{otherwise} \end{cases}$$

Prove that $\Phi_n(x) = \prod_{d|n} \left(x^{\frac{n}{d}} - 1\right)^{\mu(d)}$.

(4) Compute $\Phi_{10}(x)$ and $\Phi_{12}(x)$.

Problem 3.– Let L/K be a finite Galois extension. For $\alpha \in L$, prove that $L = K(\alpha)$ if and only if $\mathrm{Stab}_{\mathsf{G}(L/K)}(\alpha) = \{\mathrm{Id}\}$.

Problem 4.– Let K be a field of characteristic p, such that K contains a primitive n^{th} root of unity. Prove that p does not divide n.

Problem 5.– Show that $x^p - x - a \in \mathbb{F}_p[x]$ is irreducible for every $a \in \mathbb{F}_p^\times$.

Problem 6.– Give an example of a finite field extension L/K such that there are infinitely many intermediate extensions $K \subset E \subset L$.

Problem 7.– Let $K = \mathbb{F}_p(\lambda)$ and $L = \mathbb{F}_p(\mu)$ where $\mu^p = \lambda$. Show that, for $\alpha \in L$:

$$\mathsf{N}_{L/K}(\alpha) = \alpha^p, \qquad \mathrm{Tr}_{L/K}(\alpha) = 0.$$

Problem 8.– Let E/K be a finite extension. Show that E/K is separable if and only if $E \times E \to K$, $(x, y) \mapsto \mathrm{Tr}_{E/K}(xy)$ is a non–degenerate, symmetric K–bilinear form. Show that the above condition is equivalent to the existence of $a \in E$ such that $\mathrm{Tr}_{E/K}(a) \neq 0$.

In Problems 9 and 10 below, $m \in \mathbb{Z}_{\geq 1}$, K is a field, and we assume that K contains a primitive m^{th} root of unity. Thus, $\mu_m(K) \cong \mathbb{Z}/m\mathbb{Z}$.

Problem 9.– Let $a \in K^\times$ and define:

$$r = \mathrm{Min}\{\ell : a^\ell = z^m, \text{ for some } z \in K^\times\}.$$

Note that r divides m. Let $B = \{z \in K^\times : z^{m/r} = a\}$.

(1) Prove that for every $b \in B$, $x^r - b \in K[x]$ is irreducible.

(2) Show that $x^m - a = \prod_{b \in B} x^r - b$.

(3) Let L/K be the splitting extension of $x^m - a$. Prove that $\mathsf{G}(L/K)$ is cyclic of size r.

Problem 10.– Let G be a cyclic group of size m. Show that we have a (not natural) isomorphism:

$$\mathrm{Hom}_{\mathrm{gp}}(G, \mu_m(K)) \cong G.$$

(Hint: an isomorphism can be written upon choosing a primitive m^{th} root of unity $\zeta \in \mu_m(K)$ and a generator σ of G.)