LECTURE 0

(0.0) Overview of the course.– This course is divided into three components: *Category theory, Homological algebra* and *Galois theory.*

Category theory. In this part, we will go over the basic definitions of categories, functors, natural transformation of functors etc. This formalism emerged around 1945, from the works of algebraic topologists, most notably Eilenberg and MacLane. The purpose of this part is to familiarise ourselves with this language, which is extensively used in many areas of mathematics. It will also feature prominantly in the second part of the course.

Homological algebra. The main object of study for this part will be the category of R–modules, where R is a (unital) commutative ring. We will go over the abstract definition of *derived functors* and introduce Ext and Tor functors as derived of Hom and $\otimes$. Many interesting cohomology theories in mathematics can be viewed as Ext functors on an appropriate category. Tor functors play a crucial role in intersection theory (you can look up *Serre intersection multiplicity formula* for details).

Galois theory. This part of the course is not related to the other two. It also have a very extensive history, which we will go over when we start this topic. The central characters of this story are fields and their extensions.

In this lecture, we will define categories and functors, and go over several examples of these.

(0.1) Categories.– A category $\mathcal{C}$ consists of the following data:

- A class $\mathrm{Ob}(\mathcal{C})$, called *objects* of $\mathcal{C}$.
- For any two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, called *morphisms* from X to Y.
- For $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$, a map:
$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$
called *composition of morphisms.* The composition is denoted, as usual, by $(f, g) \mapsto g \circ f$.
- For every object $X \in \mathrm{Ob}(\mathcal{C})$, a distinguished morphism $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$, called *the identity morphism.*

This data is subject to the following two axioms.

(1) *Composition is associative.* That is, for every $X, Y, Z, W \in \mathrm{Ob}(\mathcal{C})$, and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Hom}_{\mathcal{C}}(Y, Z)$ and $h \in \mathrm{Hom}_{\mathcal{C}}(Z, W)$, we have:
$$(h \circ g) \circ f = h \circ (g \circ f).$$

(2) *Identity morphisms are neutral with respect to composition.* That is, for every $X, Y \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, we have:

$$f \circ \mathrm{Id}_X = f = \mathrm{Id}_Y \circ f.$$

(0.2) Examples.–

(1) **Sets**. The objects of this category are sets: that is, $X \in \mathrm{Ob}(\mathbf{Sets})$ just means that X is a set. For any two sets X, Y, $\mathrm{Hom}_{\mathbf{Sets}}(X, Y)$ consists of usual set maps $X \to Y$. $\mathrm{Id}_X : X \to X$ is the usual identity map, and composition of morphisms is defined in the natural way: $(g \circ f)(x) = g(f(x))$.

(2) Similar to the previous example, we have the following categories, familiar from your previous algebra course.
- The category of groups, **Gps**, whose objects are groups and morphisms are group homomorphisms.
- The category of abelian groups, **Ab**.
- The category of rings, **Rings**.
- The category of commutative rings, **CommRings**.
- For a ring R, we have the category of (left) R–modules, denoted by R–mod. Often, we will denote by mod–R, the category of right R–modules.
- For a field K, let $\mathbf{Vect}_K$ denote the category of K–vector spaces. Its objects are vector spaces over K, and morphisms are K–linear maps.

(3) Let **Top** denote the category of topological spaces. Its objects are topological spaces and morphisms are continuous maps.

Thus, many (if not all) structures in mathematics form a category. For instance, one has the category of differentiable manifolds, algebraic varieties etc.

(0.3) Notation.– We will often write $X \in \mathcal{C}$ instead of $X \in \mathrm{Ob}(\mathcal{C})$, to mean that X is an object of a category $\mathcal{C}$. Morphisms are often written as $f : X \to Y$, or $X \xrightarrow{f} Y$, to mean that $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

(0.4) Injective and surjective morphisms.– Let $\mathcal{C}$ be a category and $f : X \to Y$ be a morphism between two objects X, Y of $\mathcal{C}$. We say f is *injective* (or *monomorphism*) if, for every $Z \in \mathcal{C}$, the following map of sets is one-to-one:

$$\mathrm{Hom}_{\mathcal{C}}(Z, X) \to \mathrm{Hom}_{\mathcal{C}}(Z, Y), \qquad h \mapsto f \circ h.$$

In other words, f is injective if it can be cancelled from the right. That is, $f \circ h_1 = f \circ h_2 \Rightarrow h_1 = h_2$.

Similarly, we say that f is *surjective* (or, *epimorphism*) if, for every $Z \in \mathcal{C}$, the following map of sets is one-to-one:

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z), \qquad g \mapsto g \circ f.$$

In other words, f is surjective if it can be cancelled from the left. That is, $g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2$.

If f is both injective and surjective, then we say that f is a *bijection.*

Remark. In order not to cause any confusion, we will agree to use the terms *injective, surjective, bijective* in accordance with what is defined above. Many categories we will encounter will have objects which are sets together with some additional structure (see Examples (0.2)). For a morphism $f : X \to Y$ in such a category, the corresponding *purely set–theoretical notions* will be called *one-to-one, onto and both.* It is important to keep this distinction in mind, as the following example illustrates.

Example. Consider the category **Rings** of unital rings (morphisms are also assumed to be unital). Let $f : \mathbb{Z} \to \mathbb{Q}$ be the natural inclusion of the ring of integers into the field of rational numbers. Set–theoretically, this morphism is one-to-one, but not onto. However, *category–theoretically*, f is a bijection. To see this, we observe that f is clearly injective (prove this for yourself: one-to-one implies injective). To prove surjectivity, let $g_1, g_2 : \mathbb{Q} \to R$ be two morphisms such that $g_1 \circ f = g_2 \circ f$. We have to prove that $g_1 = g_2$. Since their restriction to $\mathbb{Z} \subset \mathbb{Q}$ is the same, and $g_i(1/n) = g_i(n)^{-1}$ $(i = 1, 2)$, for every $n \in \mathbb{Z}_{\neq 0}$, we get the following, for every $a/b \in \mathbb{Q}$:

$$g_1(a/b) = g_1(a)g_1(b)^{-1} = g_2(a)g_2(b)^{-1} = g_2(a/b).$$

(0.5) Left and right inverses.– Again, let $f : X \to Y$ be a morphism in a category $\mathcal{C}$. We say that f admits a left inverse, if there exists $r : Y \to X$, such that $r \circ f = \mathrm{Id}_X$. Usually, this morphism r is called a *retraction.*

Similarly, f is said to admit a right inverse, if there exists $s : Y \to X$ such that $f \circ s = \mathrm{Id}_Y$. This morphism s is often called a *section.*

If f admits both a left and a right inverse, then we say that f is an *isomorphism.*

Lemma. *Let $\mathcal{C}$ be a category and $f : X \to Y$ a morphism in $\mathcal{C}$.*

(1) *Assume that there exists $g : Y \to Y'$ such that $g \circ f$ is injective. Then f is injective.*
(2) *If f admits a left inverse, then it is injective.*
(3) *Assume that there exists $h : X' \to X$ such that $f \circ h$ is surjective. Then f is surjective.*
(4) *If f admits a right inverse, then it is surjective.*

Proof. (1). Let $Z \in \mathcal{C}$ and let $h_1, h_2 : Z \to X$ be two morphisms, such that $f \circ h_1 = f \circ h_2$. In order to prove that f is injective, we have to show that $h_1 = h_2$. Using associativity of the composition map, we get:

$$(g \circ f) \circ h_1 = g \circ (f \circ h_1) = g \circ (f \circ h_2) = (g \circ f) \circ h_2.$$

Since $g \circ f$ is assumed to be injective, this implies that $h_1 = h_2$, by definition of an injective morphism.

(2). Let $r : Y \to X$ be the left inverse of f. In (1), take $Y' = X$ and $g = r$, so that $g \circ f = \mathrm{Id}_X$ is injective. Hence, by (1) above, we get that f is injective.

(3) and (4) are proved in exactly the same way.

□

(0.6) Remark I.– Left (and right) inverses need not be unique. For instance, let K be a field, and consider the linear map $f : K^2 \to K$ given by $f(a, b) = a + b$. This map admits many sections, for instance $s_1 : K \to K^2$ given by $s_1(x) = (x, 0)$, and $s_2 : K \to K^2$ given by $s_2(x) = (0, x)$ both satisfy $f \circ s_1 = f \circ s_2 = \mathrm{Id}_K$.

However, if both left and right inverses exist, then they are unique and equal to each other. You must have seen a proof of this statement for groups, which proceeds as follows. Let $f : X \to Y$ be a morphism which admits both left and right inverses. Let $r : Y \to X$ be such that $r \circ f = \mathrm{Id}_X$. Assume there are two sections $s_1, s_2 : Y \to X$ such that $f \circ s_1 = f \circ s_2$. Then (this is the same argument as the one given in the proof of Lemma (0.5)):

$$s_1 = \mathrm{Id}_X \circ s_1 = (r \circ f) \circ s_1 = r \circ (f \circ s_1) = r \circ (f \circ s_2) = (r \circ f) \circ s_2 = \mathrm{Id}_X \circ s_2 = s_2.$$

This proves the uniqueness of right inverse. Similar argument shows the uniqueness of left inverse. Finally, if r is the left inverse and s is the right iverse of f, then:

$$r = r \circ \mathrm{Id}_Y = r \circ f \circ s = \mathrm{Id}_X \circ s = s.$$

(0.7) Remark II.– Lemma (0.5) implies that every isomorphism is a bijection. The converse is not true, as Example (0.4) above demonstrates. Here is another example.

Let $\mathcal{F}(\mathbf{Ab})$ denote the category of filtered abelian groups. That is,

- An object of $\mathcal{F}(\mathbf{Ab})$ is a descending chain:

$$F_\bullet = (F_0 \supset F_1 \supset \cdots), \text{ where } F_i \text{ is an abelian group } \forall i \in \mathbb{Z}_{\geq 0}.$$

- A morphism $f_\bullet : F_\bullet \to G_\bullet$ is a homomorphism of abelian groups $f_0 : F_0 \to G_0$ such that $f_0(F_i) \subset G_i$ for every $i \geq 0$.

Consider the following two objects of this category:

$$A_\bullet = (\mathbb{Z} \supset 0 \cdots), \qquad \text{and} \qquad B_\bullet = (\mathbb{Z} \supset 2\mathbb{Z} \supset 0 \supset \cdots).$$

Let $i : A_\bullet \to B_\bullet$ be identity on $A_0 = B_0 = \mathbb{Z}$. Then i is a bijection (prove this!). However, it does not admit either left or right inverse, since $\mathrm{Hom}_{\mathcal{F}(\mathbf{Ab})}(B_\bullet, A_\bullet) = \{0\}$.

(0.8) Functors.– Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ is the following assignment:

- For every object $X \in \mathcal{C}$, we have $F(X) \in \mathcal{D}$.
- For every morphism $f : X \to Y$ in $\mathcal{C}$, we have $F(f) : F(X) \to F(Y)$ in $\mathcal{D}$.

This assignment is required to satisfy two conditions:

(1) For every $X \in \mathcal{C}$, $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$.
(2) For every $X, Y, Z \in \mathcal{C}$ and $f : X \to Y$, $g : Y \to Z$, we have:

$$F(g \circ f) = F(g) \circ F(f).$$

A *contravariant functor* $G : \mathcal{C} \to \mathcal{D}$ differs from a covariant one, in only that it reverses the arrows of morphisms. In more detail, G is the following assignment:

- For every object $X \in \mathcal{C}$, we have $G(X) \in \mathcal{D}$.
- For every morphism $f : X \to Y$ in $\mathcal{C}$, we have $G(f) : G(Y) \to G(X)$ in $\mathcal{D}$.

This assignment is similarly assumed to satisfy: $G(\mathrm{Id}_X) = \mathrm{Id}_{G(X)}$ and $G(g \circ f) = G(f) \circ G(g)$.

(0.9) Example I: Forgetful functor.– Let $\mathcal{C}$ be any one of the categories given in Examples (0.2). Then we have $F : \mathcal{C} \to \mathbf{Sets}$ which sends each object to the underlying set (forgetting any additional structure). Such functors are often called *forgetful functors.* These functors are covariant.

(0.10) Example II: Hom **functors.–** Let $\mathcal{C}$ be an arbitrary category. Let $X \in \mathcal{C}$. We can define two functors h_X and h^X from $\mathcal{C}$ to **Sets** as follows.

- $h_X : \mathcal{C} \to \mathbf{Sets}$. It maps any $Y \in \mathcal{C}$ to the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. Every morphism $f : Y_1 \to Y_2$ in $\mathcal{C}$ gives rise to a set map (composition with f), $h_X(f) : \mathrm{Hom}_{\mathcal{C}}(X, Y_1) \to \mathrm{Hom}_{\mathcal{C}}(X, Y_2)$, given by: $g \mapsto f \circ g$. It is therefore a covariant functor.

- $h^X : \mathcal{C} \to \mathbf{Sets}$. It maps any $Y \in \mathcal{C}$ to the set $\mathrm{Hom}_{\mathcal{C}}(Y, X)$. Every morphism $f : Y_1 \to Y_2$ in $\mathcal{C}$ gives rise to a set map (again, composition with f), $h^X(f) : \mathrm{Hom}_{\mathcal{C}}(Y_2, X) \to \mathrm{Hom}_{\mathcal{C}}(Y_1, X)$, given by: $g \mapsto g \circ f$. It is therefore a contravariant functor.

These functors are sometimes denoted as $h_X = \mathrm{Hom}_{\mathcal{C}}(X, -)$ and $h^X = \mathrm{Hom}_{\mathcal{C}}(-, X)$, and called covariant and contravariant Hom–functors respectively.

A special case of h^X is the duality functor on the category $\mathbf{Vect}_K$ of K–vector spaces, where K is a field. $D : \mathbf{Vect}_K \to \mathbf{Vect}_K$ is given by:

$$D(V) = V^* = \mathrm{Hom}_K(V, K).$$

Recall that for every K–linear map $f : V \to W$, we have its transpose $f^* : W^* \to V^*$ which agrees with the definition of $h^K(f)$ given above.

(Note: in $\mathbf{Vect}_K$, the set of morphisms $\mathrm{Hom}_K(V, W)$ has an additional structure of a K–vector space. That is why the Hom–functors defined above take value in $\mathbf{Vect}_K$, not just in **Sets**).

(0.11) Example III. Fundamental group.–[1] Let $\mathbf{Top}_*$ be the category of pointed topological spaces. That is, an object of this category is a pair (X, x_0), where X is a topological space and $x_0 \in X$. A morphism in this category $(X, x_0) \to (Y, y_0)$ is a continuous map $f : X \to Y$ such that $f(x_0) = y_0$.

Recall that $\pi_1(X, x_0)$ is defined as the group of loops (up to homotopy) in X based at x_0. The assigment $(X, x_0) \mapsto \pi_1(X, x_0)$ is a covariant functor $\mathbf{Top}_* \to \mathbf{Gps}$.

[1] Optional