LECTURE 1

(1.0) Last lecture.– Recall that in the previous lecture we defined categories. We went over the definition of injective and surjective morphisms in an abstract category, and highlighted how these notions differ from the usual set–theoretic 1-1 and onto maps. We ended the last lecture with the definition of covariant and contravariant functors, and saw a few examples.

In today's lecture, we will study *natural transformations of functors*. This will allow us to understand when two categories are *equivalent*.

(1.1) Some comments.– Functors can be composed, just like functions. Thus, if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are two functors, then we get $G \circ F : \mathcal{C} \to \mathcal{E}$. With respect to composition, the *variance* of functors behaves as parity:

$$\text{Covariant} \circ \text{Covariant} = \text{Covariant}, \qquad \text{Covariant} \circ \text{Contravariant} = \text{Contravariant}$$
$$\text{Contravariant} \circ \text{Covariant} = \text{Contravariant}, \qquad \text{Contravariant} \circ \text{Contravariant} = \text{Covariant}$$

Opposite category. Often to fix ideas, people agree to talk only about covariant functors. This is achieved by introducing (artificially) the opposite of a category, by reversing the direction of the morphisms. More explicitly, if $\mathcal{C}$ is a category, we can define a new category, denoted by $\mathcal{C}^{\text{op}}$, whose objects are same as the objects of $\mathcal{C}$, and

$$\text{Hom}_{\mathcal{C}^{\text{op}}}(X, Y) := \text{Hom}_{\mathcal{C}}(Y, X)$$

Thus contravariant functors from $\mathcal{C}$ to $\mathcal{D}$ are same as covariant functors from $\mathcal{C}^{\text{op}}$ to $\mathcal{D}$.

If nothing is specified, a functor is assumed to be covariant.

(1.2) Faithful and full functors.– Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Recall that, for every $X, Y \in \mathcal{C}$, F defines a set map

$$\text{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{F_{X,Y}} \text{Hom}_{\mathcal{D}}(F(X), F(Y)),$$

given by $f \mapsto F(f)$. We say that F is *faithful*, if $F_{X,Y}$ is 1-1 map. Similarly, we say F is *full*, if $F_{X,Y}$ is onto.

Example. Consider the forgetful functor F : **Gps** $\to$ **Sets**. Since, for any two groups G and H, $F_{G,H}$ is the natural inclusion of $\text{Hom}_{\mathbf{Gps}}(G, H) \subset \text{Hom}_{\mathbf{Sets}}(G, H)$, we see that F is faithful. It is clearly not full.

Let **Top** be the category of topological space, where morphisms are continuous maps. Let **HTop** denote the category whose objects are topological spaces, and morphisms are continuous maps *up to homotopy*. (Recall that, two contiuous maps $f, g : X \to Y$ are said to be homotopic, if there exists $\widetilde{f} : X \times [0, 1] \to Y \times [0, 1]$ such that $\widetilde{f}(x, 0) = f(x)$ and $\widetilde{f}(x, 1) = g(x)$). The natural functor **Top** $\to$ **HTop** is not faithful, but full.

(1.3) Natural transformations.– Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and let $F, G : \mathcal{C} \to \mathcal{D}$ be two functors.

A *natural transformation* from F to G, denoted by η, assigns to each $X \in \mathcal{C}$, a morphism $\eta_X : F(X) \to G(X)$ in $\mathcal{D}$.

$$X \in \mathcal{C} \qquad \rightsquigarrow \qquad \eta_X \in \mathrm{Hom}_{\mathcal{D}}(F(X), G(X)).$$

This assignment is required to satisfy the following property: for any morphism $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

We say that η is a *natural isomorphism* if for every $X \in \mathcal{C}$, the morphism η_X is an isomorphism is $\mathcal{D}$.

Remarks.

(1) If both F and G are contravariant, then the last diagram needs to be modified, by flipping the direction of the vertical arrows. We do not have a notion of natural transformation between a covariant and a contravariant functor.

(2) In many category theory text, a natural transformation is depicted by a double arrow, as follows.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "F"{name=U}] \arrow[r, bend right=50, "G"'{name=D}] & \mathcal{D} \arrow[Rightarrow, from=U, to=D, "\eta"] \end{tikzcd}$$

(1.4) Example of the double-dual.– Let K be a field, and $\mathbf{Vect}_K$ be the category of K–vector spaces. Let $\mathbb{D} : \mathbf{Vect}_K \to \mathbf{Vect}_K$ be the functor of taking the dual twice: $\mathbb{D}(V) = V^{**}$. (Recall that $V^* = \mathrm{Hom}_K(V, K)$).

Recall (from Algebra I) that, for every $V \in \mathbf{Vect}_K$, there is a natural K–linear map, which we are going to denote by η_V, from V to V^{**}. This map, evaluated on a vector $v \in V$, is given by:

$$\eta_V(v)(\xi) = \xi(v) \text{ for every } \xi \in V^*.$$

Proposition. *η is a natural transformation from the identity functor* $\mathrm{Id} : \mathbf{Vect}_K \to \mathbf{Vect}_K$ *to the double-duality functor* $\mathbb{D}$.

Proof. Assume that we are given a K–linear map $f : V \to W$. Recall that, it induces $f^{**} : V^{**} \to W^{**}$, which is given as follows. If $\psi \in V^{**}$ and $\xi \in W^*$, then

$$f^{**}(\psi) : \xi \mapsto \psi(f^*(\xi))$$

(Recall that $f^*(\xi) \in V^*$ is given by $v \mapsto \xi(f(v))$.)

We need to prove that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "\eta_V"] \arrow[d, "f"'] & V^{**} \arrow[d, "f^{**}"] \\ W \arrow[r, "\eta_W"'] & W^{**} \end{tikzcd}$$

So, let $v \in V$, and let us unfold the definitions to see explicitly what $\eta_W(f(v))$ and $f^{**}(\eta_V(v))$ are, as elements of W^{**}, that is, linear forms on W^*. We will evaluate both of them on a typical element $\phi \in W^*$, and see that we get the same element of K.

By definition of η_W, we have $\eta_W(f(v)) : \phi \mapsto \phi(f(v))$. Similarly, from the definition of f^{**} and η_V written above, we get:

$$f^{**}(\eta_V(v)) : \phi \mapsto (\eta_V(v))((f^*(\phi))) = (f^*(\phi))(v) = \phi(f(v)).$$

Hence, $\eta_W(f(v)) = f^{**}(\eta_V(v)) \in W^{**}$, for every $v \in V$, which is exactly what we wanted to show. □

Remark. This $\eta : \mathrm{Id} \Rightarrow \mathbb{D}$ is not a natural isomorphism. However, if we restrict to $\mathbf{Vect}_K^{fd}$, the category of *finite–dimensional* K–vector spaces, then it is.

(1.5) Example of Hom**–functors.**– Recall (Example (0.10) of Lecture 0, page 5) that given a category $\mathcal{C}$ and an object $X \in \mathcal{C}$, we have two functors $h_X = \mathrm{Hom}_{\mathcal{C}}(X, -)$ (covariant) and $h^X = \mathrm{Hom}_{\mathcal{C}}(-, X)$ (contravariant) from $\mathcal{C}$ to **Sets**.

Every morphism $f : X \to Y$ in $\mathcal{C}$ gives rise to a natural transformation, denoted by h_f, from the functor h_Y to h_X as follows.

$$Z \in \mathcal{C} \qquad \rightsquigarrow \qquad h_{f,Z} = - \circ f : h_Y(Z) = \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z) = h_X(Z).$$

Exercise: Prove that h_f is a natural transformation of functors.

Similarly, f defines a natural transformation $h^f : h^X \Rightarrow h^Y$.

(1.6) Equivalence of categories.– A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be an *equivalence of categories* if there is a functor $G : \mathcal{D} \to \mathcal{C}$ and natural *isomorphisms* of functors

$$\phi : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F \qquad \text{and} \qquad \psi : \mathrm{Id}_{\mathcal{D}} \Rightarrow F \circ G.$$

Remark. The notion of equivalence is weaker than requiring that both compositions $G \circ F$ and $F \circ G$ are the identity functors (see Example (1.7) below).

Theorem. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if, and only if, F is faithful, full and the following condition holds:*

For every $Y \in \mathcal{D}$, there exists some $X \in \mathcal{C}$ such that $F(X)$ and Y are isomorphic in $\mathcal{D}$ (in this case, we say that F is essentially surjective or dense).

The forward implication of this theorem is proved in (1.8) below. The converse will be proved in the next lecture.

(1.7) Example.– Let K be a field and let us define a category $\mathbf{Mat}_K$ as follows. Objects of $\mathbf{Mat}_K$ are non–negative integers. For $n, m \in \mathbb{Z}_{\geq 0}$, we define

$$\mathrm{Hom}_{\mathbf{Mat}_K}(n, m) := \mathrm{Mat}_{m\times n}(K)\ m \times n \text{ matrices with entries from } K.$$

Composition of morphisms in this category is the usual matrix multiplication and the identity matrix plays the role of the identity morphism.

Let $F : \mathbf{Mat}_K \to \mathbf{Vect}_K^{fd}$ be the functor which sends n to K^n. On morphisms, F sends an $m \times n$ matrix to the linear map it defines $K^n \to K^m$.

Theorem (1.6) then implies that F is an equivalence of categories.

(1.8) Proof of Theorem (1.7): forward implication.– Assume that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories. That is, we have $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms of functors $\phi : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\psi : \mathrm{Id}_{\mathcal{D}} \Rightarrow F \circ G$.

F is essentially surjective. This is clear since for any $Y \in \mathcal{D}$, we are given an isomorphism $\psi_Y : Y \xrightarrow{\sim} F(G(Y))$.

F is faithful. For $X_1, X_2 \in \mathcal{C}$, we have isomorphisms $\phi_i : X_i \xrightarrow{\sim} G(F(X_i))$. Consider the following composition of set maps (it sends $f \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2)$ to $G(F(f))$).

$$\mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \to \mathrm{Hom}_{\mathcal{D}}(F(X_1), F(X_2)) \to \mathrm{Hom}_{\mathcal{C}}(G(F(X_1)), G(F(X_2))).$$

Since ϕ is a natural transformation of functors, for any $f \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2)$, the following diagram commutes:

$$\begin{tikzcd} X_1 \arrow[r, "\phi_1"] \arrow[d, "f"'] & G(F(X_1)) \arrow[d, "G(F(f))"] \\ X_2 \arrow[r, "\phi_2"'] & G(F(X_2)) \end{tikzcd}$$

That is, $G(F(f)) = \phi_2 \circ f \circ \phi_1^{-1}$ (recall both ϕ_1 and ϕ_2 are isomorphisms). Hence, the composition of the set maps written above $f \mapsto G(F(f))$ is a bijection (with inverse given by $g \mapsto \phi_2^{-1} \circ g \circ \phi_1$). This implies that $f \mapsto F(f)$ is injective, that is, F is faithful.

F is full. The proof of this part is similar to the one given above. Again $X_1, X_2 \in \mathcal{C}$, and we use the isomorphisms ($i = 1, 2$):

$$\phi_i : X_i \xrightarrow{\sim} G(F(X_i)) \qquad \text{and} \qquad \psi_i : F(X_i) \xrightarrow{\sim} F(G(F(X_i)))$$

We aim to show that that the following composition of set maps is a bijection.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F(X_1), F(X_2)) \arrow[r] \arrow[d, dashed] & \mathrm{Hom}_{\mathcal{C}}(G(F(X_1)), G(F(X_2))) \arrow[d] \\
\mathrm{Hom}_{\mathcal{D}}(F(X_1), F(X_2)) & \mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \arrow[l]
\end{tikzcd}$$

where,

- The two horizontal arrows are obtained from the functors G and F.
- The right vertical arrow is the inverse of the bijection considered in the proof of faithfulness. That is, $g \mapsto \phi_2^{-1} \circ g \circ \phi_1$.

You should check that the composition (depicted by the dotted arrow in the diagram above) is given by:

$$g \mapsto F(\phi_2)^{-1} \circ \psi_2 \circ g \circ \psi_1^{-1} \circ F(\phi_1)$$

and hence is a bijection. Thus, the bottom horizontal arrow $f \mapsto F(f)$ in the diagram above must be onto. This proves that F is full.