LECTURE 2

(2.0) Review.– So far we have defined: categories, functors, bijections and isomorphisms, faithful and full functors, natural transformation (and isomorphism) of functors and what it means for a functor to be an equivalence of categories. We stated the following theorem (see Theorem (1.6)):

Theorem. *$F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if, and only if:*

1. *F is faithful and full. That is, for every $X, Y \in \mathcal{C}$ the set map $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is a bijection.*
2. *For every $Y \in \mathcal{D}$, there exists $X \in \mathcal{C}$ and an isomorphism $\beta_Y : Y \xrightarrow{\sim} F(X)$ in $\mathcal{D}$. This condition is often stated: F is essentially surjective, or dense.*

We proved the forward implication of this theorem in §1.8. Let us start by proving the converse.

(2.1) Proof of Theorem 1.6: reverse implication.– Assume that F is faithful, full and essentially surjective. In order to prove the reverse implication of the theorem, we need to construct a functor $G : \mathcal{D} \to \mathcal{C}$, and two natural isomorphisms

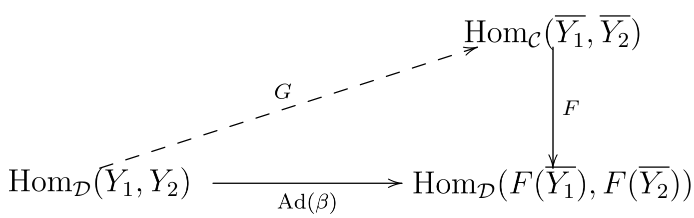
$$\phi : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F \qquad \text{and} \qquad \psi : \mathrm{Id}_{\mathcal{D}} \Rightarrow F \circ G.$$

For every $Y \in \mathcal{D}$, choose $\overline{Y} \in \mathcal{C}$ and an isomorphism (exists by essential surjectivity of F) $\beta_Y : Y \xrightarrow{\sim} F(\overline{Y})$. Note that this allows us to define a set bijection

$$\mathrm{Ad}(\beta) : \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F(\overline{Y_1}), F(\overline{Y_2}))$$

given by $g \mapsto \beta_{Y_2} \circ g \circ \beta_{Y_1}^{-1}$..

Define the functor $G : \mathcal{D} \to \mathcal{C}$ on objects as: $G(Y) = \overline{Y}$. For morphisms, $G : \mathrm{Hom}_{\mathcal{D}}(Y_1, Y_2) \to \mathrm{Hom}_{\mathcal{C}}(\overline{Y_1}, \overline{Y_2})$ is defined to be $F^{-1} \circ \mathrm{Ad}(\beta)$:

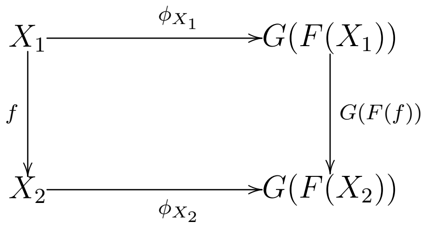


Here, we are using that F sets up a bijection between $\mathrm{Hom}_{\mathcal{C}}(\overline{Y_1}, \overline{Y_2})$, and $\mathrm{Hom}_{\mathcal{D}}(F(\overline{Y_1}), F(\overline{Y_2}))$, and F^{-1} denotes its inverse. It is easy to see that it is a functor: $G(\mathrm{Id}_Y) = \mathrm{Id}_{\overline{Y}}$, and $G(g_1 \circ g_2) = G(g_1) \circ G(g_2)$.

Constructon of ϕ. For every $X \in \mathcal{C}$, ϕ_X is defined to be $F^{-1}(\beta_{F(X)})$. That is, the inverse image of $\beta_{F(X)} \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(\overline{F(X)}))$ under the bijection defined by F:

$$F : \mathrm{Hom}_{\mathcal{C}}(X, \overline{F(X)}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F(X), F(\overline{F(X)})).$$

We have to show that for every morphism $f : X_1 \to X_2$ in $\mathcal{C}$, the following diagram commutes:



By definition,

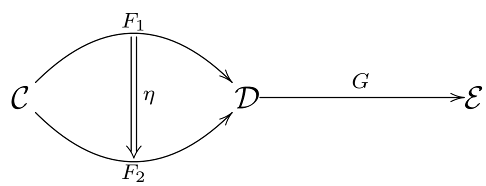
$$G(F(f)) = F^{-1}\left(\beta_{F(X_2)} \circ F(f) \circ \beta_{F(X_1)}^{-1}\right) = \phi_{X_2} \circ f \circ \phi_{X_1}^{-1},$$

as required.

The construction of ψ is similar, and is left as an exercise.

(2.2) Remarks.–

1. Functors and natural transformations can also be "composed" in the following sense. Let $F_1, F_2 : \mathcal{C} \to \mathcal{D}$ be two functors, $G : \mathcal{D} \to \mathcal{E}$ another, and η be a natural transformation from F_1 to F_2.



Then, we get a natural transformation $G(\eta) : G \circ F_1 \Rightarrow G \circ F_2$, defined by:

$$X \in \mathcal{C} \qquad \rightsquigarrow \qquad G(\eta_X) : G(F_1(X)) \to G(F_2(X)).$$

Similarly, if $F : \mathcal{C} \to \mathcal{D}$ and $G_1, G_2 : \mathcal{D} \to \mathcal{E}$ are functors and μ is a natural transformation from G_1 to G_2, we get a natural transformation μ_F from $G_1 \circ F$ to $G_2 \circ F$ by:

$$X \in \mathcal{C} \qquad \rightsquigarrow \qquad \mu_{F(X)} : G_1(F(X)) \to G_2(F(X)).$$

2. (Product categories and functors of "several variables"). Given two categories $\mathcal{C}_1$ and $\mathcal{C}_2$, we can define *the product category* $\mathcal{C}_1 \times \mathcal{C}_2$ as follows:

- Objects of $\mathcal{C}_1 \times \mathcal{C}_2$ are pairs (X_1, X_2) where $X_1 \in \mathcal{C}_1$ and $X_2 \in \mathcal{C}_2$.

- For (X_1, X_2) and (Y_1, Y_2) two objects in $\mathcal{C}_1 \times \mathcal{C}_2$, the set of morphisms is defined by:

$$\mathrm{Hom}_{\mathcal{C}_1 \times \mathcal{C}_2}((X_1, X_2), (Y_1, Y_2)) := \mathrm{Hom}_{\mathcal{C}_1}(X_1, Y_1) \times \mathrm{Hom}_{\mathcal{C}_2}(X_2, Y_2).$$

The composition is defined component–wise, and $\mathrm{Id}_{(X_1,X_2)} = (\mathrm{Id}_{X_1}, \mathrm{Id}_{X_2})$.

A functor $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ can then be thought of as a functor of 2 variables.

(2.3) Example.– Let K be a field and consider the category $\mathbf{Vect}_K$ of K–vector spaces. Consider the following two functors.

- $F_1 = (-)^* \otimes - : \mathbf{Vect}_K \times \mathbf{Vect}_K \to \mathbf{Vect}_K$. That is, $F_1(V, W) = V^* \otimes W$.

- $F_2 = \mathrm{Hom}_K(-, -) : \mathbf{Vect}_K \times \mathbf{Vect}_K \to \mathbf{Vect}_K$. That is, $F_2(V, W) = \mathrm{Hom}_K(V, W)$.

Note that these functors are "mixed", that is, covariant in the second variable and contravariant in the first.

Recall (from Algebra I), for any two $V, W \in \mathbf{Vect}_K$, we have a natural K–linear map:

$$\eta_{V,W} : V^* \otimes W \to \mathrm{Hom}_K(V, W),$$

given by: $\eta_{V,W}(\xi \otimes w) : v \mapsto \xi(v) \cdot w$, for every $\xi \in V^*, v \in V$ and $w \in W$. (Verify that this is a natural transformation).

Recall (also from Algebra I) that, if we restrict F_1 and F_2 to $\mathbf{Vect}_K^{fd} \times \mathbf{Vect}_K$, then η is a natural isomorphism.

(2.4) Adjoint functors.– Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and let there be given two functors $\mathcal{C} \underset{F_2}{\overset{F_1}{\rightleftarrows}} \mathcal{D}$. We say that (F_1, F_2) is *a pair of adjoint functors* (or, F_1 *is left adjoint of* F_2, or, F_2 *is right adjoint of* F_1), if, for every $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, we are given bijections which are natural in X and Y:

$$\beta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, F_2(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F_1(X), Y).$$

Note that *naturality* in X and Y means the following: for every morphism $f : X \to X'$ in $\mathcal{C}$ and $g : Y' \to Y$ in $\mathcal{D}$, the following diagram commutes:

$$\begin{array}{ccccc}
F_2(g)\circ\alpha\circ f \in & \mathrm{Hom}_{\mathcal{C}}(X, F_2(Y)) & \xrightarrow{\beta_{X,Y}} & \mathrm{Hom}_{\mathcal{D}}(F_1(X), Y) & \ni g\circ\gamma\circ F_1(f) \\
\uparrow & \uparrow & & \uparrow & \uparrow \\
\alpha \in & \mathrm{Hom}_{\mathcal{C}}(X', F_2(Y')) & \xrightarrow[\beta_{X',Y'}]{} & \mathrm{Hom}_{\mathcal{D}}(F_1(X'), Y') & \ni \gamma
\end{array}$$

In other words, β is a natural isomorphism between the following two functors from $\mathcal{C} \times \mathcal{D} \to \mathbf{Sets}$:

$$\begin{array}{ccccc}
 & & \mathcal{C}\times\mathcal{C} & & \\
 & \nearrow^{\mathrm{Id}_{\mathcal{C}}\times F_2} & & \searrow^{\mathrm{Hom}_{\mathcal{C}}(-,-)} & \\
\mathcal{C}\times\mathcal{D} & & \Downarrow \beta & & \mathbf{Sets} \\
 & \searrow_{F_1\times \mathrm{Id}_{\mathcal{D}}} & & \nearrow_{\mathrm{Hom}_{\mathcal{D}}(-,-)} & \\
 & & \mathcal{D}\times\mathcal{D} & &
\end{array}$$

Remark. Adjointness of functors is a generalization of equivalence (see §2.6 below), and plays an important role in many areas of mathematics. In representation theory, *induction and restriction* functors form an adjoint pair. In topology and algebraic geometry *pull–back and push–forward* functors form an adjoint pair.

(2.5) Example I. Left adjoint to the forgetful functor.– Let $F : \mathbf{Gps} \to \mathbf{Sets}$ be the forgetful functor. Let $\mathrm{Free} : \mathbf{Sets} \to \mathbf{Gps}$ be the functor which sends a set X to the free group generated by X, denoted here by $\mathrm{Free}(X)$.

Recall that the free group is defined by a universal property, which states that, for any group H, $\mathrm{Hom}_{\mathbf{Sets}}(X, H)$ is same as $\mathrm{Hom}_{\mathbf{Gps}}(\mathrm{Free}(X), H)$. That is, for every $X \in \mathbf{Sets}$ and $H \in \mathbf{Gps}$, we have bijections:

$$\mathrm{Hom}_{\mathbf{Sets}}(X, F(H)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Gps}}(\mathrm{Free}(X), H).$$

You should verify that these bijections are natural in X and H. Thus, we conclude that (Free, F) forms an adjoint pair.

(2.6) Example II. Equivalence to adjointness.– Assume that $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories. Thus, we have $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphism $\phi : \mathrm{Id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\psi : \mathrm{Id}_{\mathcal{D}} \Rightarrow F \circ G$.

Lemma. *(F, G) and (G, F) are both adjoint pairs.*

Proof. Let us prove this lemma for the pair (F, G) (proof for (G, F) is entirely analoguous and omitted here). For every $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, we need to construct natural bijections $\beta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F(X), Y)$. This is done by employing the natural isomorphism $\psi_Y : Y \xrightarrow{\sim} F(G(Y))$. For any morphism $h : X \to G(Y)$ in $\mathcal{C}$, define

$$\beta_{X,Y}(h) := \psi_Y^{-1} \circ F(h) \in \mathrm{Hom}_{\mathcal{D}}(F(X), Y).$$

Now we prove that our definition is natural in X and Y. See §2.4 for what it means, and the diagram whose commutativity we need to check.

Thus, given $f : X \to X'$ in $\mathcal{C}$, and $g : Y' \to Y$ in $\mathcal{D}$, we have to verify the following the following equation for every $\alpha \in \mathrm{Hom}_{\mathcal{C}}(X', G(Y'))$:

$$\beta_{X,Y}\left(G(g) \circ \alpha \circ f\right) = g \circ \beta_{X',Y'}(\alpha) \circ F(f).$$

Using our definition of β, the left–hand side of this equation becomes:

$$\text{L.H.S.} = \psi_Y^{-1} \circ F(G(g)) \circ F(\alpha) \circ F(f).$$

And the right–hand side becomes:

$$\text{R.H.S.} = g \circ \psi_{Y'}^{-1} \circ F(\alpha) \circ F(f).$$

Note that by naturality of ψ, we have

$$\psi_Y \circ g = F(G(g)) \circ \psi_{Y'},$$

which proves that the left and the right–hand sides are equal, as claimed. □

(2.7) Example III: Induction and restriction functors.– Let G be a group and $H < G$ be a subgroup. Let us define the category G-**Sets** as follows:

- An object of this category is a set together with a G–action.
  Recall that this means: an object is a pair (X, α), where X is a set and $\alpha : G \times X \to X$ is a set map satisfying two conditions:

(1) $\alpha(e,x) = x$, for every $x \in X$. Here, $e \in G$ is the identity element of G.
(2) $\alpha(gh,x) = \alpha(g,\alpha(h,x))$ for every $g,h \in G$ and $x \in X$.

As is customary, we will suppress α from the notation and simply write $g \cdot x = \alpha(g,x)$.

- A morphism is a set map which commutes with G–action. More explicitly, a G–morphism from X_1 to X_2 is a set map $f : X_1 \to X_2$ such that $f(g \cdot x) = g \cdot f(x)$ for every $g \in G$ and $x \in X$.

Since $H < G$, we get a functor, called restriction (from G to H) functor:

$$\mathrm{Res}_{\mathrm{H}}^{\mathrm{G}} : G\text{-}\mathbf{Sets} \to H\text{-}\mathbf{Sets}.$$

This functor admits a left adjoint, called induction functor. It is defined as follows. Let Y be a set with an H–action. As a set $\mathrm{Ind}_H^G(Y)$ is defined as the set of equivalence classes in $G \times Y$, modulo the following equivalence relation:

$$(gh, y) \sim (g, h \cdot y) \text{ for every } g, h \in G \text{ and } y \in Y.$$

This set has a natural G–action given by left–multiplication on the first component. It is also denoted by $G \times_H Y$ in the literature.

Exercise. Verify that $\mathrm{Ind}_H^G : H\text{-}\mathbf{Sets} \to G\text{-}\mathbf{Sets}$ is a functor, and $(\mathrm{Ind}_H^G, \mathrm{Res}_H^G)$ forms an adjoint pair.