LECTURE 3

(3.0) Review.– Recall that last time we defined *adjoint functors*. Given two categories $\mathcal{C}$ and $\mathcal{D}$ and two functors

$$\mathcal{C} \underset{F_2}{\overset{F_1}{\rightleftarrows}} \mathcal{D}$$

we say (F_1, F_2, β) (or, just (F_1, F_2)) is an adjoint pair, if for every $X \in \mathcal{C}$ and $Y \in \mathcal{D}$, we are given bijections:

$$\beta_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, F_2(Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F_1(X), Y)$$

which are natural in X and Y.

Naturality means the following. Given a morphism $f : X \to X'$ in $\mathcal{C}$ and $g : Y' \to Y$ in $\mathcal{D}$, the following equation must hold (both sides are in $\mathrm{Hom}_{\mathcal{D}}(F_1(X), Y)$):

$$(1) \qquad \beta_{X,Y}\left(F_2(g) \circ \alpha \circ f\right) = g \circ \beta_{X',Y'}(\alpha) \circ F_1(f), \ \forall\, \alpha \in \mathrm{Hom}_{\mathcal{C}}(X', F_2(Y')).$$

(see the commutative diagram on page 3 of Lecture 2).

Today we will discuss another way to define an adjoint pair, using the *unit* and *counit* of adjunction.

(3.1) Unit and counit of adjunction.– The first part of the following lemma was stated as Problem 12 in Homework 1.

Lemma. *Let (F_1, F_2, β) be an adjoint pair of functors. Then there are natural transformations of functors:*

$$\varepsilon : F_1 \circ F_2 \Rightarrow \mathrm{Id}_{\mathcal{D}} \qquad \textit{and} \qquad \eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow F_2 \circ F_1$$

called the counit and unit of adjunction respectively. Furthermore, the following compositions of natural transformations are both identity:

$$F_1 \xRightarrow{F_1(\eta)} F_1F_2F_1 \xRightarrow{\varepsilon_{F_1}} F_1$$

$$F_2 \xRightarrow{\eta_{F_2}} F_2F_1F_2 \xRightarrow{F_2(\varepsilon)} F_2$$

Recall the definition of "composition" of functors and natural transformations from §2.2 (1). For instance, $F_1(\eta)$ sends each object $X \in \mathcal{C}$ to the morphism $F_1(\eta_X) : F_1(X) \to F_1F_2F_1(X)$.

Proof. We begin by constructing the natural transformations ε and η. For an object $Y \in \mathcal{D}$, define $\varepsilon_Y : F_1(F_2(Y)) \to Y$ as $\beta_{F_2(Y),Y}(\mathrm{Id}_{F_2(Y)})$. That is, the image of $\mathrm{Id}_{F_2(Y)}$ under the given bijection:

$$\beta_{F_2(Y),Y} : \mathrm{Hom}_{\mathcal{C}}(F_2(Y), F_2(Y)) \to \mathrm{Hom}_{\mathcal{D}}(F_1(F_2(Y)), Y).$$

Similarly, given an object $X \in \mathcal{C}$, define $\eta_X : X \to F_2(F_1(X))$ as $\beta^{-1}_{X,F_1(X)}(\mathrm{Id}_{F_1(X)})$.

We need to prove that ε and η thus defined are in fact natural transformations. I highly recommend to try to prove it for yourself (i.e, using equation (1), prove the required one for ε and η), before reading the proof given below. We will only prove the naturality of η, leaving the same for ε as an exercise.

Naturality of η. Assume we are given a morphism $a : X_1 \to X_2$. We need to prove the commuatitivity of the following diagram:

$$\begin{array}{ccc} X_1 & \xrightarrow{\eta_{X_1}} & F_2(F_1(X_1)) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle F_2(F_1(a))} \\ X_2 & \xrightarrow{\eta_{X_2}} & F_2(F_1(X_2)) \end{array}$$

or, in equations: $\eta_{X_2} \circ a = F_2(F_1(a)) \circ \eta_{X_1}$, for every $a \in \mathrm{Hom}_{\mathcal{C}}(X_1, X_2)$. We begin by considering the following two diagrams, which commute by naturality of β:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X_1, F_2(F_1(X_2))) & \xrightarrow{\beta_{X_1,F_1(X_2)}} & \mathrm{Hom}_{\mathcal{D}}(F_1(X_1), F_1(X_2)) \\ {\scriptstyle -\circ a}\uparrow & & \uparrow{\scriptstyle -\circ F_1(a)} \\ \mathrm{Hom}_{\mathcal{C}}(X_2, F_2(F_1(X_2))) & \xrightarrow{\beta_{X_2,F_1(X_2)}} & \mathrm{Hom}_{\mathcal{D}}(F_1(X_2), F_1(X_2)) \end{array}$$

computing the image of $\eta_{X_2} \in \mathrm{Hom}_{\mathcal{C}}(X_2, F_2(F_1(X_2)))$, we get

$$\beta_{X_1,F_1(X_1)}(\eta_{X_2} \circ a) = F_1(a). \tag{2}$$

Note that this equation is obtained from (1) upon substituting: $X = X_1$, $X' = X_2$, $f = a$, $Y = Y' = F_1(X_2)$, $g = \mathrm{Id}_{F_1(X_2)}$ and $\alpha = \eta_{X_2}$.

The second diagram is as follows:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X_1, F_2(F_1(X_2))) & \xrightarrow{\beta_{X_1,F_1(X_2)}} & \mathrm{Hom}_{\mathcal{D}}(F_1(X_1), F_1(X_2)) \\ {\scriptstyle F_2(F_1(a))\circ -}\uparrow & & \uparrow{\scriptstyle F_1(a)\circ -} \\ \mathrm{Hom}_{\mathcal{C}}(X_1, F_2(F_1(X_1))) & \xrightarrow{\beta_{X_1,F_1(X_1)}} & \mathrm{Hom}_{\mathcal{D}}(F_1(X_1), F_1(X_1)) \end{array}$$

following the element $\eta_{X_1} \in \mathrm{Hom}_{\mathcal{C}}(X_1, F_2(F_1(X_1)))$, we get:

$$\beta_{X_1,F_1(X_2)}(F_2(F_1(a)) \circ \eta_{X_1}) = F_1(a). \tag{3}$$

Comparing (2) and (3), and using the fact that $\beta_{X_1,F_1(X_2)}$ is a bijection, we get the desired equation $\eta_{X_2} \circ a = F_2(F_1(a)) \circ \eta_{X_1}$.

Let us now prove that the following composition of natural transformations is identity:

$$F_1 \xRightarrow{F_1(\eta)} F_1F_2F_1 \xRightarrow{\varepsilon_{F_1}} F_1$$

That is, given an object $X \in \mathcal{C}$, we have to show that $\varepsilon_{F_1(X)} \circ F_1(\eta_X) = \mathrm{Id}_{F_1(X)}$. For this, we consider the following diagram:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(X, F_2(F_1(X))) & \xrightarrow{\beta_{X,F_1(X)}} & \mathrm{Hom}_{\mathcal{D}}(F_1(X), F_1(X)) \\ \uparrow{\scriptstyle -\circ\eta_X} & & \uparrow{\scriptstyle -\circ F_1(\eta_X)} \\ \mathrm{Hom}_{\mathcal{C}}(F_2F_1(X), F_2F_1(X)) & \xrightarrow{\beta_{F_2F_1(X),F_1(X)}} & \mathrm{Hom}_{\mathcal{D}}(F_1F_2F_1(X), F_1(X)) \end{array}$$

Following the identity morphism $\mathrm{Id}_{F_2F_1(X)} \in \mathrm{Hom}_{\mathcal{C}}(F_2F_1(X), F_2F_1(X))$ in two ways, gives us the following equation:

$$\beta_{X,F_1(X)}(\eta_X) = \beta_{F_2F_1(X),F_1(X)}(\mathrm{Id}_{F_2F_1(X)}) \circ F_1(\eta_X).$$

Note that the left–hand side is $\mathrm{Id}_{F_1(X)}$ by definition of η_X, and the right–hand side is $\varepsilon_{F_1(X)} \circ F_1(\eta_X)$ by definition of ε.

□

(3.2) Examples.– Let us consider the example of the adjoint pair:

$$\mathbf{Sets} \underset{F}{\overset{\mathrm{Free}}{\rightleftarrows}} \mathbf{Gps}$$

where F is the forgetful functor and Free sends a set X to the free group generated by X.

The counit of this adjunction $\varepsilon : \mathrm{Free} \circ F \Rightarrow \mathrm{Id}_{\mathbf{Gps}}$ is the following group homomorphism, for any group G:

$$\varepsilon_G : \mathrm{Free}(G) \to G.$$

Here $\mathrm{Free}(G)$ is the free group on letters $\{x_g : g \in G\}$, and ε_G is the unique group homomorphism extending $\varepsilon(x_g) = g$ for every $g \in G$.

Similarly, the unit $\eta : \mathrm{Id}_{\mathbf{Sets}} \Rightarrow F \circ \mathrm{Free}$ is the following set map, for a given set X:

$$\eta_X : X \to \mathrm{Free}(X),$$

which sends each $x \in X$ to the corresponding generator of the free group $\mathrm{Free}(X)$.

(3.3) Converse of Lemma (3.1).– The converse of the construction given above also holds. That is, assume we are given:

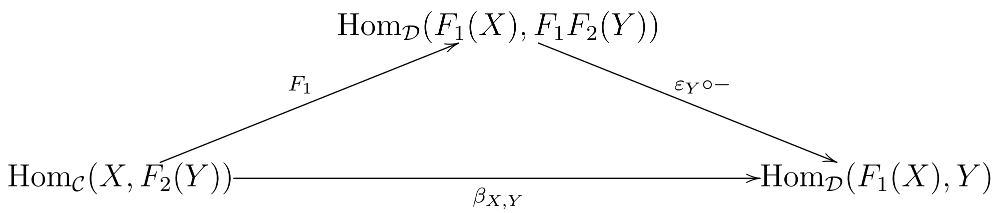
(1) A pair of functors $F_1 : \mathcal{C} \to \mathcal{D}$ and $F_2 : \mathcal{D} \to \mathcal{C}$.
(2) Two natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow F_2F_1$ and $\varepsilon : F_1F_2 \Rightarrow \mathcal{D}$.

Further, we assume that the following two compositions of natural transformations are identity:

$$F_1 \xRightarrow{F_1(\eta)} F_1F_2F_1 \xRightarrow{\varepsilon_{F_1}} F_1$$

$$F_2 \xRightarrow{\eta_{F_2}} F_2F_1F_2 \xRightarrow{F_2(\varepsilon)} F_2$$

We can use this data to define $\beta_{X,Y}$ by the following diagram:



Lemma. (F_1, F_2) *forms an adjoint pair, with* β *defined as above.*

Proof. The diagram given above realizes β as a composition of natural transformations of functors from $\mathcal{C} \times \mathcal{D}$ to **Sets**:

- F_1 viewed as a natural transformation from $\mathrm{Hom}_{\mathcal{C}}(-, F_2(-))$ to $\mathrm{Hom}_{\mathcal{D}}(F_1(-), F_1F_2(-))$.
- $\varepsilon \circ -$ viewed as natural transformation from $\mathrm{Hom}_{\mathcal{D}}(F_1(-), F_1F_2(-))$ to $\mathrm{Hom}_{\mathcal{D}}(F_1(-), -)$.

Hence β is a natural transformation. Now we prove that $\beta_{X,Y}$ is a bijection for every $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Recall that $\beta_{X,Y}(f) = \varepsilon_Y \circ F_1(f)$, for every $f : X \to F_2(Y)$.

Let us construct another natural map in the reverse direction: $\gamma_{X,Y} : \mathrm{Hom}_{\mathcal{D}}(F_1(X), Y) \to \mathrm{Hom}_{\mathcal{C}}(X, F_2(Y))$ given by: $\gamma_{X,Y}(g) = F_2(g) \circ \eta_X$. We now claim that $\beta_{X,Y}$ and $\gamma_{X,Y}$ are inverse to each other.

$\beta_{X,Y} \circ \gamma_{X,Y} = \mathrm{Id}$. Given a morphism $g \in \mathrm{Hom}_{\mathcal{D}}(F_1(X), Y)$, we have:

$$\beta_{X,Y}(\gamma_{X,Y}(g)) = \beta_{X,Y}(F_2(g) \circ \eta_X) = \varepsilon_Y \circ F_1(F_2(g) \circ \eta_X) = \varepsilon_Y \circ F_1F_2(g) \circ F_1(\eta_X).$$

Note that, by naturality of ε, we have $\varepsilon_Y \circ F_1F_2(g) = g \circ \varepsilon_{F_1(X)}$. This implies:

$$\beta_{X,Y}(\gamma_{X,Y}(g)) = \varepsilon_Y \circ F_1F_2(g) \circ F_1(\eta_X) = g \circ \varepsilon_{F_1(X)} \circ F_1(\eta_X) = g \circ \mathrm{Id}_{F_1(X)} = g.$$

Here, we have used the fact that $F_1 \Rightarrow F_1F_2F_1 \Rightarrow F_1$ is the identity natural transformation, i.e, for every $X \in \mathcal{C}$, $\varepsilon_{F_1(X)} \circ F_1(\eta_X) = \mathrm{Id}_{F_1(X)}$.

$\gamma_{X,Y} \circ \beta_{X,Y} = \mathrm{Id}$. Similarly, for every $f \in \mathrm{Hom}_{\mathcal{C}}(X, F_2(Y))$, we can carry out the same calculation, using naturality of η and identity of $F_2 \Rightarrow F_2F_1F_2 \Rightarrow F_2$, i.e, $F_2(\varepsilon_Y) \circ \eta_{F_2(Y)} = \mathrm{Id}_{F_2(Y)}$.

$$\begin{aligned}\gamma_{X,Y}(\beta_{X,Y}(f)) &= F_2(\varepsilon_Y \circ F_1(f)) \circ \eta_X = F_2(\varepsilon_Y) \circ F_2F_1(f) \circ \eta_X \\ &= F_2(\varepsilon_Y) \circ \eta_{F_2(Y)} \circ f = \mathrm{Id}_{F_2(Y)} \circ f = f.\end{aligned}$$

□