LECTURE 4

(4.0) Goal of this lecture.– This lecture is devoted to proving *Yoneda embedding theorem*. This theorem allows us to define *representable functors*, and pose the question of *representability* which is equivalent to defining a mathematical object via its *universal property*.

(4.1) Categories of functors.– Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. We define two categories $\mathcal{F}(\mathcal{C},\mathcal{D})$ and $\mathcal{F}(\mathcal{C}^{\mathrm{op}},\mathcal{D})$ as follows.

$\mathcal{F}(\mathcal{C},\mathcal{D})$. Objects of this category are all covariant functors $\mathcal{C}\to\mathcal{D}$. Given two objects F,G in this category, morphisms are natural transformations from F to G:

$$\mathrm{Hom}_{\mathcal{F}(\mathcal{C},\mathcal{D})}(F,G) := \{\eta : F\Rightarrow G \text{ a natural transformation}\}.$$

Id_F is the identity natural transformation, and composition of natural transformation is defined *pointwise*: for $\eta : F\Rightarrow G$ and $\mu : G\Rightarrow H$, we have $\mu\circ\eta : F\Rightarrow H$ given by:

$$X\in\mathcal{C} \qquad \rightsquigarrow \qquad F(X)\xrightarrow{\mu_X\circ\eta_X} H(X).$$

It is straightforward to verify that this is a category.

Similarly, $\mathcal{F}(\mathcal{C}^{\mathrm{op}},\mathcal{D})$ is the category whose objects are all contravariant functors $\mathcal{C}\to\mathcal{D}$, with morphisms being natural transformations as above.

(4.2) Yoneda embedding.– Given a category $\mathcal{C}$, and an object $X\in\mathcal{C}$, recall the construction of $\mathsf{h}_X\in\mathcal{F}(\mathcal{C},\mathcal{D})$ and $\mathsf{h}^X\in\mathcal{F}(\mathcal{C}^{\mathrm{op}},\mathcal{D})$ from §0.10. That is, for every $Y\in\mathcal{C}$:

$$\mathsf{h}_X(Y)=\mathrm{Hom}_{\mathcal{C}}(X,Y) \qquad \text{and} \qquad \mathsf{h}^X(Y)=\mathrm{Hom}_{\mathcal{C}}(Y,X).$$

For every morphism $g : Y_1\to Y_2$ in $\mathcal{C}$, we have $\mathsf{h}_X(g)=g\circ -$ and $\mathsf{h}^X(g)=-\circ g$.

Similarly, given a morphism $f : X_1\to X_2$, we defined natural transformations $\mathsf{h}_f : \mathsf{h}_{X_2}\Rightarrow\mathsf{h}_{X_1}$ and $\mathsf{h}^f : \mathsf{h}^{X_1}\Rightarrow\mathsf{h}^{X_2}$ in §1.5, as follows:

$$Y\in\mathcal{C} \qquad \rightsquigarrow \qquad \begin{array}{c} \mathsf{h}_{f,Y} = -\circ f : \mathrm{Hom}_{\mathcal{C}}(X_2,Y)\to\mathrm{Hom}_{\mathcal{C}}(X_1,Y) \\ \mathsf{h}^f_Y = f\circ - : \mathrm{Hom}_{\mathcal{C}}(Y,X_1)\to\mathrm{Hom}_{\mathcal{C}}(Y,X_2) \end{array}$$

The fact that h_f and h^f are natural transformations was left as an exercise in §1.5. Let us prove it here for h^f.

Lemma. h^f *is a natural transformation. Moreover, we have*

(1) $\mathsf{h}^{\mathrm{Id}_X}=\mathrm{Id}_{\mathsf{h}^X}$. *Here both sides are natural transformations* $\mathsf{h}^X\Rightarrow\mathsf{h}^X$.

(2) $\mathsf{h}^{f'\circ f}=\mathsf{h}^{f'}\circ\mathsf{h}^f$, *for any two morphisms* $f : X\to X'$ *and* $f' : X'\to X''$. *Again, both sides of the claimed equation are natural transformations* $\mathsf{h}^X\Rightarrow\mathsf{h}^{X''}$.

Proof. Let us begin by proving that h^f is a natural transformation. That is, given any $g : Y_1 \to Y_2$, we need to prove the commutativity of the following diagram (recall that h^X is contravariant):

$$\begin{array}{ccc} \mathsf{h}^{X_1}(Y_2) & \xrightarrow{\mathsf{h}^f_{Y_2}} & \mathsf{h}^{X_2}(Y_2) \\ {\scriptstyle \mathsf{h}^{X_1}(g)}\downarrow & & \downarrow{\scriptstyle \mathsf{h}^{X_2}(g)} \\ \mathsf{h}^{X_1}(Y_1) & \xrightarrow[\mathsf{h}^f_{Y_1}]{} & \mathsf{h}^{X_2}(Y_1) \end{array}$$

So, given a morphism $\alpha \in \mathsf{h}^{X_1}(Y_2) = \mathrm{Hom}(Y_2, X_1)$, we can compute the two compositions:

$$(\mathsf{h}^{X_2}(g) \circ \mathsf{h}^f_{Y_2})(\alpha) = \mathsf{h}^{X_2}(g)(f \circ \alpha) = (f \circ \alpha) \circ g$$
$$(\mathsf{h}^f_{Y_1} \circ \mathsf{h}^{X_1}(g))(\alpha) = \mathsf{h}^f_{Y_1}(\alpha \circ g) = f \circ (\alpha \circ g)$$

The two morphisms on the right–hand sides above are equal, by associativity of composition.

Proof of (1). The natural transformation $\mathsf{h}^{\mathrm{Id}_X}$, on an arbitrary object $Y \in \mathcal{C}$ gives rise to the following set map:

$$\mathsf{h}^{\mathrm{Id}_X}_Y : \mathrm{Hom}(Y, X) \to \mathrm{Hom}(Y, X) \qquad g \mapsto \mathrm{Id}_X \circ g = g.$$

Hence, $\mathsf{h}^{\mathrm{Id}_X}$ is equal to $\mathrm{Id}_{\mathsf{h}^X}$.

Proof of (2). Given $f : X \to X'$ and $f' : X' \to X''$, let us compute $\mathsf{h}^{f' \circ f}_Y : \mathsf{h}^X(Y) \to \mathsf{h}^{X''}(Y)$. By definition, it is given by: $\mathrm{Hom}(Y, X) \ni g \mapsto (f' \circ f) \circ g = f' \circ (f \circ g)$. The last composition is equal to $(\mathsf{h}^{f'} \circ \mathsf{h}^f)(g)$, and the claimed identity follows. □

The statement of this lemma can be summarized as follows.

Corollary. *We have a covariant functor* $\mathsf{h}^\bullet : \mathcal{C} \to \mathcal{F}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$*:*

$$X \in \mathcal{C} \quad \rightsquigarrow \quad \mathsf{h}^X \in \mathcal{F}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$$
$$f : X_1 \to X_2 \quad \rightsquigarrow \quad \mathsf{h}^f : \mathsf{h}^{X_1} \Rightarrow \mathsf{h}^{X_2}$$

Similarly, there is a contravariant functor $\mathsf{h}_\bullet : \mathcal{C} \to \mathcal{F}(\mathcal{C}, \mathbf{Sets})$.

(4.3) Embedding theorem.– The functors $\mathsf{h}^\bullet$ and $\mathsf{h}_\bullet$ are often called Yoneda embeddings. We state and prove below why the are *embeddings*.

Theorem. *Let $\mathcal{C}$ be a category. Then the functors $\mathsf{h}^\bullet$ and $\mathsf{h}_\bullet$ are faithful and full.*

Proof. Let us prove this theorem for $\mathsf{h}^\bullet : \mathcal{C} \to \mathcal{F}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$. Recall that, by definition of faithful and full, what we need to prove that the following map of sets is a bijection:

$$\mathrm{Hom}_{\mathcal{C}}(X_1, X_2) \to \mathrm{Hom}_{\mathcal{F}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})}(\mathsf{h}^{X_1}, \mathsf{h}^{X_2}), \qquad f \mapsto \mathsf{h}^f.$$

We will prove a more general statement below, which implies this. Namely, let $F \in \mathcal{F}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$.

Claim. For any $X \in \mathcal{C}$, we have a bijection between $F(X)$ and $\mathrm{Hom}_{\mathcal{F}(\mathcal{C}^{\mathrm{op}},\mathbf{Sets})}(\mathsf{h}^X, F)$, given by:

$$F(X) \ni a \mapsto \eta(a) : h^X \Rightarrow F,$$

where, $\eta(a)_Y : \mathrm{Hom}(Y, X) \to F(Y)$ sends a morphism $g : Y \to X$, to $F(g)(a)$. (Recall that F is contravariant, so $F(g) : F(X) \to F(Y)$).

You should check it for yourself that the theorem follows from this claim, by taking $X = X_1$ and $F = \mathsf{h}^{X_2}$ (it is almost trivial, the only small thing you need to verify is that $\eta(f)$ defined here for $f \in \mathsf{h}^{X_2}(X_1) = \mathrm{Hom}(X_1, X_2)$ is same as h^f given in §4.2 above).

Proof of the Claim. We will construct a map of sets $\mu : \mathrm{Hom}_{\mathcal{F}(\mathcal{C}^{\mathrm{op}},\mathbf{Sets})}(\mathsf{h}^X, F) \to F(X)$ and prove that $\mu \circ \eta = \mathrm{Id}_{F(X)}$ and $\eta \circ \mu = \mathrm{Id}_{\mathrm{Hom}_{\mathcal{F}(\mathcal{C}^{\mathrm{op}},\mathbf{Sets})}(\mathsf{h}^X,F)}$.

So, given a natural transformation $\xi : \mathsf{h}^X \Rightarrow F$, evaluate it on $X \in \mathcal{C}$, $\xi_X : \mathrm{Hom}_{\mathcal{C}}(X, X) \to F(X)$, and define:

$$\mu(\xi) := \xi_X(\mathrm{Id}_X) \in F(X).$$

$\mu \circ \eta$ *is identity.* Given $a \in F(X)$, let us compute $\mu(\eta(a))$. By definition, it is given by:

$$\mu(\eta(a)) = \eta(a)_X(\mathrm{Id}_X) = F(\mathrm{Id}_X)(a) = \mathrm{Id}_{F(X)}(a) = a.$$

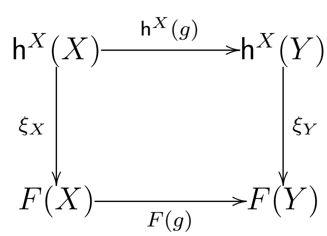
$\eta \circ \mu$ *is identity.* Now, given a natural transformation $\xi : \mathsf{h}^X \Rightarrow F$, consider the natural transformation $\eta(\mu(\xi)) : \mathsf{h}^X \Rightarrow F$. Evaluating on an object $Y \in \mathcal{C}$, we obtain:

$$\eta(\mu(\xi))_Y : \mathrm{Hom}_{\mathcal{C}}(Y, X) \to F(Y),$$

sends a morphism $g : Y \to X$, to $F(g)(\mu(\xi))$. By definition of μ, we have:

$$F(g)(\mu(\xi)) = F(g)(\xi_X(\mathrm{Id}_X)).$$

By naturality of ξ, we have $F(g) \circ \xi_X = \xi_Y \circ \mathsf{h}^X(g)$ (see the commutative diagram below)



Therefore, we obtain (using the fact that $\mathsf{h}^X(g) = - \circ g$):

$$\eta(\mu(\xi))_Y(g) = F(g)(\xi_X(\mathrm{Id}_X)) = \xi_Y(\mathsf{h}^X(g)(\mathrm{Id}_X)) = \xi_Y(g).$$

Hence, $\eta(\mu(\xi)) = \xi$ for every $\xi \in \mathrm{Hom}_{\mathcal{F}(\mathcal{C}^{\mathrm{op}},\mathbf{Sets})}(\mathsf{h}^X, F)$, and the claim follows. □

(4.4) Consequence of Theorem (4.3).– Yoneda embedding theorem realizes $\mathcal{C}$ as a subcategory of $\mathcal{F}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$. The following is a direct corollary of this theorem:

Corollary. *Let $X_1, X_2 \in \mathcal{C}$ and assume that there is a natural isomorphism of functor $\eta : \mathsf{h}^{X_1} \Rightarrow \mathsf{h}^{X_2}$. Then, there is a unique isomorphism $f : X_1 \to X_2$ such that $\eta = \mathsf{h}^f$.*

Similarly, if there is a natural isomorphism of functors $\xi : \mathsf{h}_{X_1} \Rightarrow \mathsf{h}_{X_2}$, then there is a unique isomorphism $f : X_2 \to X_1$ such that $\mathsf{h}_f = \xi$.

Proof. By Theorem (4.3), we know that $\eta = \mathsf{h}^f$ for a unique morphism $f : X_1 \to X_2$ in $\mathcal{C}$. Similarly, if $\mu : \mathsf{h}^{X_2} \Rightarrow \mathsf{h}^{X_1}$ is the inverse of η in $\mathcal{F}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$ category, then $\mu = \mathsf{h}^g$ for a unique morphism $g : X_2 \to X_1$. Now $\mathrm{Id}_{\mathsf{h}^{X_1}} = \mu \circ \eta = \mathsf{h}^{g \circ f}$, by Lemma (4.2) part (2), and $\mathrm{Id}_{\mathsf{h}^{X_1}} = \mathsf{h}^{\mathrm{Id}_{X_1}}$ by Lemma (4.2) part (1). Hence, $\mathsf{h}^{g\circ f} = \mathsf{h}^{\mathrm{Id}_{X_1}}$, and by Theorem (4.3), this implies $g \circ f = \mathrm{Id}_{X_1}$. Similarly we can show that $f \circ g = \mathrm{Id}_{X_2}$, proving that f is an isomorphism. $\square$

(4.5) Representable functors.– Yoneda embedding is in general not an equivalence of categories (it is faithful and full, but not essentially surjective in general). This prompts the following definition.

Definition. Let $F \in \mathcal{F}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$ be an arbitrary contravariant functor. We say that F *is representable* if there exists a pair (X, ι^X), where $X \in \mathcal{C}$ and $\iota^X : \mathsf{h}^X \Rightarrow F$ is a natural isomorphism. By a little abuse of terminology, we say that F *is represented by* X (omitting ι^X), or that X *represents* F.

Similarly, if G is a covariant functor $\mathcal{C} \to \mathbf{Sets}$, we say it is *representable* if there exists a pair (X, ι_X), where $X \in \mathcal{C}$ and $\iota_X : \mathsf{h}_X \Rightarrow G$ is a natural isomorphism.

As a consequence of Yoneda embedding theorem, we know that if a functor F is represented by an object X, then X is unique, up to a unique isomorphism.

Proof. Assume that F is a contravariant functor, represented by two pairs (X, ι^X) and $(X', \iota^{X'})$. Then, by composing $(\iota^{X'})^{-1} \circ \iota^X$, we obtain a natural isomorphism $\mathsf{h}^X \Rightarrow \mathsf{h}^{X'}$. By Corollary (4.4), there is a unique isomorphism $f : X \to X'$ such that this natural isomorphism is of the form h^f. $\square$

Remark. Almost every mathematical object defined via a *universal property* is a solution to a particular representability problem. In several areas of mathematics (especially algebraic geometry) we often first define a functor and then pose the problem of whether it is representable or not. For instance, moduli spaces, Hilbert scheme of points, classifying space of a group, Deligne–Mumford stacks and so on, are defined in this fashion.

(4.6) Unfolding the definitions.– Again, let $\mathcal{C}$ be a category and let $F \in \mathcal{F}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets})$. Assume that F is representable, by a pair (X, ι^X). Thus, $\iota^X : h^X \Rightarrow F$ is a natural isomorphism. This gives us two pieces of data (note: this is how universal properties are stated):

(1) $p_X = (\iota^X)_X(\mathrm{Id}_X) \in F(X)$. That is, *there is a distinguished element p_X of $F(X)$.*

(2) For every $Y \in \mathcal{C}$, we get a bijection $(\iota^X)_Y : \mathrm{Hom}_{\mathcal{C}}(Y, X) \to F(Y)$. Thus, every element $b \in F(Y)$ comes from a morphism $g : Y \to X$. Moreover, by naturality of

ι^X we have the following commutative diagram:

$$\begin{array}{ccc} \mathsf{h}^X(X) & \xrightarrow{(\iota^X)_X} & F(X) \\ {\scriptstyle \mathsf{h}^X(g)}\downarrow & & \downarrow{\scriptstyle F(g)} \\ \mathsf{h}^X(Y) & \xrightarrow[(\iota^X)_Y]{} & F(Y) \end{array}$$

Following the identity morphism $\mathrm{Id}_X \in \mathsf{h}^X(X)$ in two ways, we obtain $F(g)(p_X) = (\iota^X)_Y(g) = b$.

In other words, *given any $b \in F(Y)$, there exists a unique morphism $g : Y \to X$ such that $F(g)(p_X) = b$.*

(4.7) Some examples.–

I. Consider the category of groups **Gps** and fix a set X. Define a functor $F : \mathbf{Gps} \to \mathbf{Sets}$ by $F(G) = \mathrm{Hom}_{\mathbf{Sets}}(X, G)$. This is a covariant functor (since for a group homomorphism $f : G_1 \to G_2$, we get a map of sets $f \circ - : F(G_1) \to F(G_2)$). This functor is representable, and the object representing is the free group on the set X. You should convince yourself that the universal property of the free group, which you must have encountered before, is same as the two items we unfolded in the previous section.

II. Consider again the category of groups. Fix two groups G_1 and G_2 and define the functor $F : \mathbf{Gps} \to \mathbf{Sets}$ given by:

$$F(H) = \mathrm{Hom}_{\mathbf{Gps}}(H, G_1) \times \mathrm{Hom}_{\mathbf{Gps}}(H, G_2).$$

This is a contravariant functor, which is again representable. It is represented by the direct product $G_1 \times G_2$. Again, you should work it out for yourself what *universal property* the previous statement implies for $G_1 \times G_2$. For fun, think of which functor is represented by the *semi-direct product*.

III.[1] Let $\mathbf{HTop}_*$ be the category whose objects are topological spaces with a fixed base point, and morphisms are base point preserving continuous maps, up to homotopy. The fundamental group $\pi_1 : \mathbf{HTop}_* \to \mathbf{Gps}$ is a covariant, representable functor, represented by the object $(S^1, 1)$.

IV.[2] Consider the category $\mathbf{HTop}^{pc}$, whose objects are Hausdorff, paracompact spaces, and morphisms are continuous maps (again up to homotopy). Let $n \in \mathbb{Z}_{\geq 1}$ and consider the functor:

$$X \mapsto \{\text{rank } n \text{ vector bundles on } X\}/\text{isomorphisms}$$

This is a contravariant functor $\mathbf{HTop}^{pc} \to \mathbf{Sets}$. It is a (very) non–trivial theorem to prove, that this functor is representable. In fact, proving this theorem amounts to constructing a paracompact topological space, called *Grassmannian* denoted by $\mathrm{Gr}(n, \infty)$. In a way, that this space represents the functor given above is its definition.

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[1]Optional

[2]Optional. Read more about it in *J. Milnor and J. Stasheff, Characteristic classes*, Theorem 5.6.