LECTURE 5

(5.0) Review.– Recall that last time we stated and proved *Yoneda embedding theorem.* For a category $\mathcal{C}$, we constructed two functors:

$$\mathcal{C} \xrightarrow{\mathsf{h}^\bullet} \mathcal{F}(\mathcal{C}, \mathbf{Sets}),$$
$$\mathcal{C} \xrightarrow{\mathsf{h}_\bullet} \mathcal{F}(\mathcal{C}^{\mathrm{op}}, \mathbf{Sets}).$$

Theorem. *Both these functors are faithful and full.*

We defined *representable fuctors* as functors naturally isomorphic to h^X (contravariant case), or h_X (covariant case), for some $X \in \mathcal{C}$. Morover, such natural isomorphisms are assumed to be given, which we denoted by: $\iota^X : \mathsf{h}^X \to F$ (contravariant case), and $\iota_X : \mathsf{h}_X \to F$ (covariant case). We showed that there are bijections, which we denoted by μ:

$$\mathrm{Hom}_{\mathcal{F}(\mathcal{C}^{\mathrm{op}},\mathbf{Sets})}(h^X, F) \xrightarrow{\sim} F(X), \qquad \mathrm{Hom}_{\mathcal{F}(\mathcal{C},\mathbf{Sets})}(h_X, F) \xrightarrow{\sim} F(X),$$

given by $\mu(\eta) = \eta_X(\mathrm{Id}_X)$. Thus $\mu(\iota^X) = p^X \in F(X)$ determines ι^X (similarly, $\mu(\iota_X) = p_X \in F(X)$).

Since ι^X and ι_X are natural isomorphism, we have bijections, for every object $Y \in \mathcal{C}$:

$$\mathsf{h}^X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X) \xrightarrow{\iota^X_Y} F(Y), \qquad \mathsf{h}^X(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\iota_{X,Y}} F(Y)$$

Thus, we unfolded the statement (X, ι^X) *(resp.* (X, ι_X)*) represents a contravariant (resp. covariant) functor* F into the following: we are given a distinguished element $p^X \in F(X)$ (resp. $p_X \in F(X)$), such that for every $Y \in \mathcal{C}$ and $b \in F(Y)$, there is a unique morphism $f : Y \to X$ (resp. $g : X \to Y$) such that $b = F(f)(p^X)$ (resp. $b = F(g)(p_X)$). The argument above provides the forward implication, while the converse is left as an exercise - see Problem 8 of Homework 2).

Last time, we advertised the following theme:

Universal properties $\leftrightarrow$ Representability problems.

Today, we are going to study *direct sums and direct products* in an arbitrary category $\mathcal{C}$, as an example of this slogan.

(5.1) Direct sum and direct product.– We will begin with defining direct sum and product of a family (=set) of objects in a category, as solutions to a representability problem (i.e, the object satisfying a universal property). Let I be a set, $\mathcal{C}$ a category, and assume that we are given objects $X_i \in \mathcal{C}$, for every $i \in I$. Let us denote this collection by $\mathfrak{X} = \{X_i\}_{i \in I}$. Define two functors (contravariant and covariant respectively):

$$\mathsf{h}^{\mathfrak{X}} : \mathcal{C} \to \mathbf{Sets}, \qquad \mathsf{h}_{\mathfrak{X}} : \mathcal{C} \to \mathbf{Sets},$$

given by

$$\mathsf{h}^{\mathfrak{X}}(Y) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(Y, X_i), \qquad \mathsf{h}_{\mathfrak{X}}(Y) = \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y)$$

Definition. If $\mathsf{h}^{\mathfrak{X}}$ (resp. $\mathsf{h}_{\mathfrak{X}}$) is representable, then the representing object is called *direct product* (resp. *direct sum*) of $\{X_i\}_{i\in I}$, and is denoted by $\prod\limits_{i\in I}^{\mathcal{C}} X_i$ (resp. $\bigoplus\limits_{i\in I}^{\mathcal{C}} X_i$).

A remark on the notation, and other terminology. Often we will drop the superscript $\mathcal{C}$ from $\bigoplus, \prod$, if the category in question is clear from the context. It is important to remember that these notions depend on the category, for instance (see §5.4 below) direct sum of two copies of $\mathbb{Z}$ is (i) disjoint union $\mathbb{Z} \sqcup \mathbb{Z}$ in the category of sets, (ii) ordinary $\mathbb{Z}^2$ in the category of abelian groups, and (iii) $\mathrm{Free}(a, b)$ in the category of groups.

Some authors use the terms *product and coproduct* instead of *direct product and direct sum*, and use $\bigsqcup$ for $\bigoplus$ (they still use $\prod$ for (direct) product). Their notation and terminology is inspired from the category of sets, where direct sum is same as disjoint union, while direct product is the ordinary cartesian product (see §5.4).

(5.2) Universal property of direct sums.– Assume that for a family of objects $\mathfrak{X} = \{X_i\}_{i\in I}$, their direct sum exists, say denoted by (X, ι_X). Recall that this means we are given a bijection, for every $Y \in \mathcal{C}$:

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\iota_{X,Y}} \prod_{j\in I} \mathrm{Hom}_{\mathcal{C}}(X_j, Y).$$

Taking $Y = X$ we obtain $\iota_{X,X}(\mathrm{Id}_X) = (\iota_j : X_j \to X)_{j\in I}$: a collection of morphisms from $X_j \to X$, for every $j \in I$.

Moreover, these morphisms have the property that, for every $Y \in \mathcal{C}$ and $g_j : X_j \to Y$, there is a unique morphism $g : X \to Y$ such that $\iota_{X,Y}(g) = (g_j)_{j\in I}$. The commutativity of the following diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(X, X) \arrow[r, "\iota_{X,X}"] \arrow[d, "g\circ -"'] & \prod_{j\in I} \mathrm{Hom}_{\mathcal{C}}(X_j, X) \arrow[d, "(-\circ g)_{j\in I}"] \\
\mathrm{Hom}_{\mathcal{C}}(X, Y) \arrow[r, "\iota_{X,Y}"'] & \prod_{j\in I} \mathrm{Hom}_{\mathcal{C}}(X_j, Y)
\end{tikzcd}$$

implies that

$$\iota_j \circ g = g_j \text{ for every } j \in I.$$

Summarizing: $X = \bigoplus\limits_{j\in I}^{\mathcal{C}} X_j$ means the following:

(1) We are given morphisms $X_j \xrightarrow{\iota_j} X$ for every $j \in I$.
(2) For every object $Y \in \mathcal{C}$, and morphisms $g_j : X_j \to Y$, there is a unique morphism $g : X \to Y$ such that $g \circ \iota_j = g_j$ for every $j \in I$.

(5.3) Universal property of direct products.– As in the previous section, we have: $X = \prod_{i\in I}^{\mathcal{C}} X_i$ is the direct product of the family of objects $\{X_i\}_{i\in I}$ if:

(1) We are given morphisms $X \xrightarrow{p_i} X_i$, for every $i \in I$.
(2) For every object $Y \in \mathcal{C}$ and morphisms $f_i : Y \to X_i (i \in I)$, there is a unique morphism $f : Y \to X$, such that $p_i \circ f = f_i$.

(5.4) Examples.–
I. Sets : Given a family of sets $\{X_i\}_{i\in I}$, it is easy to see that:

$$\bigoplus_{i\in I}^{\textbf{Sets}} X_i = \bigsqcup_{i\in I} X_i, \qquad \prod_{i\in I}^{\textbf{Sets}} X_i = \text{Cartesian product of } X_i,$$

with the canonical morphisms are $p_i : \prod_{j\in I} X_j \to X_i$ (resp. $\iota_j : X_j \to \bigsqcup_{i\in I} X_i$) are the natural projections (resp. inclusions).

II. Ab : In the category of abelian groups, the direct sum and product are given as follows:

$$\bigoplus_{i\in I}^{\textbf{Ab}} G_i = \{(x_i \in G_i)_{i\in I} : g_i = 0 \text{ for all but finitely many } i \in I\}.$$

The idea is as follows. The ordinary cartesian product $\prod_{j\in I} G_j$ canonically contains G_j via $(\iota_j(x))_i = \delta_{ij}x$. However, given arbitrary morphisms $g_j : G_j \to H$ (another abelian group), the only way to extend it to set $g((x_i)_{i\in I}) = \sum_{i\in I} g_i(x_i)$, which is guarenteed to make sense, if all but finitely many $x_i = 0$.

It is easy to see that $\prod_{i\in I}^{\textbf{Ab}} G_i$ is just the ordinary cartesian product, with component–wise group operations.

III. R-mod : Previous example carries over verbatim to the category of left (resp. right) modules over a ring R.

IV. Gps : The direct product in this category is still same as the cartesian product. However, (showing this in general is assigned in Homework 2) for instance direct sum of two copies of $\mathbb{Z}$, in the category of all groups, is free group on 2 letters. Usually denoted by $\mathbb{Z} * \mathbb{Z} = \text{Free}(a, b)$, called the *smash product* in earlier literature.

V. Fields : We claim that direct sum does not exist in the category of fields. The reason is that we require a morphism of fields to send $1 \mapsto 1$. Thus, there are fields k_1, k_2, for instance $k_1 = \mathbb{Z}/p\mathbb{Z}$ for some prime p, and $k_2 = \mathbb{Q}$, such that the set of morphisms $\text{Hom}_{\textbf{Fields}}(k_1, k_2) =$

$\emptyset$. So, it is easy to see that the functor: $L \mapsto \mathrm{Hom}(\mathbb{Z}/p\mathbb{Z}, L) \times \mathrm{Hom}(\mathbb{Q}, L) = \emptyset$, sends every object to empty set. Thus it cannot be representable (since for a representable functor $F = \mathsf{h}_X$, $F(X) \ni \mathrm{Id}_X$ is always non–empty).

Similar argument can be given to show that direct product also does not exist in this category.

VI. $\mathbf{Vect}_K^{\mathrm{fd}}$: This example is a special case of the category R-mod. However, it is easy to see that *infinite* direct sums and products do not exist in this category, because of the finite–dimensionality restriction.