LECTURE 6

(6.0) Overview.– Recall that last time we saw the examples of direct sums and products, defined as solutions to a representability problem.

Given a category $\mathcal{C}$, an indexing set I and a collection of objects $\mathfrak{X} = \{X_i\}_{i\in I}$, their direct sum and product are defined via the data of the following natural isomorphisms:

$$\mathrm{Hom}_{\mathcal{C}}\left(\bigoplus_{i\in I}^{\mathcal{C}} X_i, -\right) \Rightarrow \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, -),$$

$$\mathrm{Hom}_{\mathcal{C}}\left(-, \prod_{i\in I}^{\mathcal{C}} X_i\right) \Rightarrow \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(-, X_i).$$

Today we will discuss a variant of this theme, called *direct and inverse limits*. In mathematical literature, these are also called *injective and projective limits* or *colimits and limits*. In this course, we will stick with direct and inverse limits.

(6.1) Direct and Inverse family of objects.– Let us assume that we are given a partially ordered set $(I, \leq)$. Recall that this means: I is a set, $\leq$ is a relation on I (a relation is a subset of $I \times I$, we write $i \leq j$ if (i, j) is in the subset). This relation is required to satisfy the following properties:

(1) $i \leq i$ for every $i \in I$.
(2) $i \leq j \leq k$ implies $i \leq k$, for every $i, j, k \in I$.
(3) $i \leq j$ and $j \leq i$ implies $i = j$.

In many textbooks, an additional axiom is imposed on this poset (short for partially ordered set).

The poset $(I, \leq)$ is *right directed* (or just *directed*) if for every $i, j \in I$, there exists a $k \in I$ such that $i \leq k$ and $j \leq k$.

I will not assume this property, by default, and will explicitly mention that we are talking about a right directed poset, if needed.

Definition. A *direct system of objects* with respect to $(I, \leq)$, valued in a category $\mathcal{C}$, consists of

- a collection of objects $\{X_i\}_{i\in I}$ of $\mathcal{C}$, and
- for every $i \leq j$, a morphism $\psi_{ji} : X_i \to X_j$,

such that (i) $\psi_{ii} = \mathrm{Id}_{X_i}$, and (ii) for every $i \leq j \leq k$ we have $\psi_{kj} \circ \psi_{ji} = \psi_{ki}$.

Similarly, an *inverse system of objects* with respect to $(I, \leq)$, valued in a category $\mathcal{C}$, consists of

- a collection of objects $\{X_i\}_{i\in I}$ of $\mathcal{C}$, and
- for every $i \leq j$, a morphism $\varphi_{ij} : X_j \to X_i$,

such that (i) $\varphi_{ii} = \mathrm{Id}_{X_i}$ and (ii) for every $i \leq j \leq k$ we have $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$.

(6.2) Direct and inverse systems as functor.– A different way to think about direct and inverse systems of objects in $\mathcal{C}$, is to define a category $\mathbf{I}$, associated to the poset $(I, \leq)$ as follows. Its objects are elements of I, and for every $i, j \in I$,

$$\mathrm{Hom}_{\mathbf{I}}(i,j) = \begin{cases} \text{Singleton } i \to j & \text{if } i \leq j, \\ \emptyset & \text{otherwise.} \end{cases}$$

Composition of morphisms is uniquely given by the partial order, and $\mathrm{Id}_i = i \to i$ is the only element in $\mathrm{Hom}_{\mathbf{I}}(i,i)$.

Now, we can easily see that

- Objects of the category of covariant functors $\mathcal{F}(\mathbf{I}, \mathcal{C})$ are *direct systems*.
- Objects of the category of contravariant functors $\mathcal{F}(\mathbf{I}^{\mathrm{op}}, \mathcal{C})$ are *inverse systems*.

The advantage of this viewpoint is that it makes the notion of *morphisms between direct/inverse systems* obvious (as morphisms in the respective category of functors).

(6.3) Direct limit.– Let $\mathfrak{X} = \{(X_i)_{i\in I}, (\psi_{ji} : X_i \to X_j)_{i\leq j}\}$ be a direct system of objects in a category $\mathcal{C}$, with respect to a poset $(I, \leq)$. Consider the following (covariant) functor:

$$\mathsf{h}_{\mathfrak{X}} : \mathcal{C} \to \mathbf{Sets},$$

given on objects by:

$$\mathsf{h}_{\mathfrak{X}}(Y) = \{(X_i \xrightarrow{f_i} Y)_{i\in I} : \text{ for every } i \leq j,\ f_j \circ \psi_{ji} = f_i\} \subset \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, Y).$$

Definition. If the above functor $\mathsf{h}_{\mathfrak{X}}$ is representable, the object representing it is called *the direct limit of the direct system* $\mathfrak{X}$. It is denoted by $\varinjlim_{(I,\leq)} X_i$, or just $\varinjlim X_i$, if the poset is clear from the context.

To spell out the universal property of direct limits, we use the same ideas as in §4.6. Namely, $X = \varinjlim_{(I,\leq)} X_i$ means the following:

- We are given morphisms $\iota_j : X_j \to X$ for every $j \in I$ such that $\iota_k \circ \psi_{kj} = \iota_j$ for every $j \leq k$.
- For every $Y \in \mathcal{C}$ and morphisms $g_j : X_j \to Y$ satisfying $g_k \circ \psi_{kj} = g_j$ for every $j \leq k$, there is a unique morphism $g : X \to Y$ such that $g \circ \iota_j = g_j$ for every $j \in I$.

(6.4) Example of direct limits in Ab.– In practice, direct limit is obtained as a "quotient" of direct sum. Let us consider the example of the category of abelian groups $\mathbf{Ab}$. The following lemma holds with the same proof in the category R-mod.

Lemma. *Direct limits exist in the category of abelian groups.*

Proof. Let $(I, \leq)$ be an arbitrary poset and let $\mathfrak{X}$ be a direct system of abelian groups: $\{G_i\}_{i\in I}$ together with morphisms $\psi_{ji} : G_i \to G_j$, for every $i \leq j$. As before, we assume that (i) $\psi_{ii} = \mathrm{Id}_{G_i}$ and (ii) for every $i \leq j \leq k$, we have $\psi_{kj} \circ \psi_{ji} = \psi_{ki}$. We begin by constructing an abelian group G, together with canonical morphisms $\iota_j : G_j \to G$. We will then show that it satisfies the universal property of the direct limit.

Consider $\widetilde{G} = \bigoplus_{i\in I} G_i$. Recall that $\widetilde{G}$ comes together with canonical morphisms $\widetilde{\iota}_j : G_j \to \widetilde{G}$, given by $(\widetilde{\iota}_j(x))_k = \delta_{jk}x$.

Let N be the subgroup of $\widetilde{G}$ generated by the following set of elements:

$$N = \langle \widetilde{\iota}_j(x) - \widetilde{\iota}_k(\psi_{kj}(x)) : \forall\ j \leq k \rangle .$$

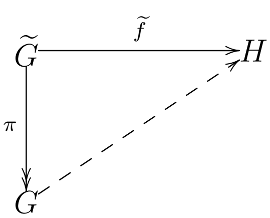
Define: $G = \widetilde{G}/N$, let $\pi : \widetilde{G} \to G$ be the natural projection map. For each $j \in I$, we have canonical morphisms $\iota_j : G_j \to G$ given as the composition $\iota_j = \pi \circ \widetilde{\iota}_j$.

First thing we have to check is: for every $j \leq k$: $\iota_k \circ \psi_{kj} = \iota_j$. This is almost by definition of N, since for every $x \in G_j$:

$$\widetilde{\iota}_k(\psi_{kj}(x)) - \widetilde{\iota}_j(x) \in N \Rightarrow \pi(\widetilde{\iota}_k(\psi_{kj}(x)) - \widetilde{\iota}_j(x)) = 0.$$

Hence $\iota_k(\psi_{kj}(x)) = \iota_j(x)$.

Now assume that we are given an abelian group H, together with morphisms $f_j : G_j \to H$ satisfying the commutativity relations: $f_k \circ \psi_{kj} = f_j$ for every $j \leq k$. Forgetting these relations for a moment, by the universal property of direct sum, we get a unique morphism $\widetilde{f} : \widetilde{G} \to H$ such that, for every $j \in I$, $\widetilde{f} \circ \widetilde{\iota}_j = f_j$. Now, all we have to show is that this morphism $\widetilde{f}$ factors through $G = \widetilde{G}/N$:



That is, $N \subset \mathrm{Ker}(\widetilde{f})$. To see this, take a typical generator $\widetilde{\iota}_j(x) - \widetilde{\iota}_k(\psi_{kj}(x))$ of N (here $j \leq k$). We have:

$$\widetilde{f}\,(\widetilde{\iota}_j(x) - \widetilde{\iota}_k(\psi_{kj}(x))) = f_j(x) - f_k(\psi_{kj}(x)) = 0,$$

since $\widetilde{f} \circ \widetilde{\iota}_j = f_j$ and $f_j = f_k \circ \psi_{kj}$. Also, for every $j \in I$: $f \circ \iota_j = f \circ \pi \circ \widetilde{\iota}_j = \widetilde{f} \circ \widetilde{\iota}_j = f_j$, as we wanted.

The uniqueness of f is a direct consequence of that of $\widetilde{f}$, and is left as a very easy exercise. □

(6.5) Example: germs of holomorphic functions.– In the theory of functions of a complex variable, one encounters the following definition.

Definition. Let $z \in \mathbb{C}$. *Germs of holomorphic functions near* z, denoted by $\mathcal{O}_z$, is the set of equivalence classes of pairs (f, U), where U is an open set containing z, and $f : U \to \mathbb{C}$ is a holomorphic function, under the following equivalence relation:

$$(f, U) \sim (g, V) \text{ if there exists an open set } W \subset U \cap V \text{ such that } f|_W \equiv g|_W\ .$$

$\mathcal{O}_z$ is in fact more than just a set, it is a commutative algebra over $\mathbb{C}$. It is even easy to see that it is *local* [1], with the unique maximal ideal given by:

$$\mathfrak{m}_z = \{[(f, U)] : f(z) = 0\}.$$

This example is generalized to what is known as *stalks of a sheaf near a point* (see the next section). We can view the ring of germs of holomorphic functions near $z \in \mathbb{C}$, as a direct limit of an appropriate direct system of commutative algebras over $\mathbb{C}$.

Consider a poset $I_{\mathbb{C}}^z$ whose elements are open sets in $\mathbb{C}$ which contain z, and the partial order is given by reverse inclusion:

$$U \leq V \text{ means } V \subset U.$$

For $U \in I_{\mathbb{C}}^z$, let $\mathcal{O}(U)$ be the ring of all holomorphic functions $f : U \to \mathbb{C}$. For every $U \leq V$, the corresponding ring homomorphism $\rho_V^U : \mathcal{O}(U) \to \mathcal{O}(V)$ is merely restriction of functions from the domain U to V.

It is easy to see that $\{(\mathcal{O}(U))_{U \in I_{\mathbb{C}}^z}, (\rho_V^U)_{U \leq U}\}$ is a direct system of commutative algebras over $\mathbb{C}$. I leave it for you to convince yourself that:

$$\mathcal{O}_z = \varinjlim_{(I_{\mathbb{C}}^z, \leq)} \mathcal{O}(U) \text{ often written as } \varinjlim_{z \in U} \mathcal{O}(U).$$

(6.6) Example: presheaves.– [2] The example of the previous section is generalized to arbitrary topological spaces, as follows. Let X be a topological space. A *presheaf of commutative rings* $\mathcal{F}$ on X is the data of:

- a commutative ring $\mathcal{F}(U)$ for every open set $U \subset X$, and
- a unital ring homomorphism, called restriction map, $\rho_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$, for every $V \subset U$.

This data is subject to the following axioms: (i) $\rho_U^U = \mathrm{Id}_{\mathcal{F}(U)}$, and (ii) for every $W \subset V \subset U$, we have $\rho_W^V \circ \rho_V^U = \rho_W^U$.

Thus, it is clear that a presheaf is nothing but a direct system, valued in an appropriate category (in the definition above, **CommRings**) with respect to the following poset I_X. The elements of I_X are open sets in X, and the partial order is the reverse inclusion, as in the

[1] A commutative ring A is said to be local, if it has a unique maximal ideal, or, equivalently, the set of all non–invertible elements of A form an ideal.

[2] First defined by French mathematician Jean Leray (1906-1998) during his time as a prisnor of war (1940-1945) in Edelbach, Austria.

previous paragraph.

Analogous to the ring of germs of holomorphic functions, we can consider a point $x \in X$, and restrict our poset I_X^x to consist of only those open sets which contain x. The direct limit of the direct system $\{(\mathcal{F}(U))_{U\in I_X^x}, (\rho_V^U)_{U\le V}\}$ is called *stalks of the presheaf* $\mathcal{F}$ at $x \in X$:

$$\mathcal{F}_x := \varinjlim_{x\in U} \mathcal{F}(U).$$

(6.7) Inverse limit.– Now let $\mathfrak{X} = \{(X_i)_{i\in I}, (\varphi_{ij} : X_j \to X_i)_{i\le j}\}$ be an inverse system of objects in a category $\mathcal{C}$, with respect to a poset $(I, \le)$. Consider the following (contravariant) functor:

$$\mathsf{h}^{\mathfrak{X}} : \mathcal{C} \to \mathbf{Sets},$$

given on objects by:

$$\mathsf{h}^{\mathfrak{X}}(Y) = \{(Y \xrightarrow{f_i} X_i)_{i\in I} : \text{ for every } i \le j,\ \varphi_{ij} \circ f_j = f_i\} \subset \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(Y, X_i).$$

Definition. If the above functor $\mathsf{h}^{\mathfrak{X}}$ is representable, the object representing it is called *the inverse limit of the inverse system* $\mathfrak{X}$. It is denoted by $\varprojlim_{(I,\le)} X_i$, or just $\varprojlim X_i$, if the poset is clear from the context.

Again, $X = \varprojlim_{(I,\le)} X_i$ means the following:

- We are given morphisms $p_j : X \to X_j$ for every $j \in I$ such that $\varphi_{jk} \circ p_k = p_j$ for every $j \le k$.
- For every $Y \in \mathcal{C}$ and morphisms $f_j : Y \to X_j$ satisfying $\varphi_{jk} \circ f_k = f_j$ for every $j \le k$, there is a unique morphism $f : Y \to X$ such that $p_j \circ f = f_j$ for every $j \in I$.