LECTURE 7

(7.0) Review.– Recall that last time we defined direct and inverse limits. Given a poset $(I, \leq)$, a direct (resp. inverse) system, valued in a category $\mathcal{C}$ can be quickly defined as an object of the functor category $\mathcal{F}(\mathbf{I}, \mathcal{C})$ (resp. $\mathcal{F}(\mathbf{I}^{\mathrm{op}}, \mathcal{C})$). Here, $\mathbf{I}$ is the category defined using the poset $(I, \leq)$ as in §6.2.

Thus, given a direct system $\{(X_i), (\psi_{ji} : X_i \to X_j)\}$, its direct limit is the unique object representing the following covariant functor $\mathcal{C} \to \mathbf{Sets}$:

$$Y \mapsto \{(f_i : X_i \to Y) : f_j \circ \psi_{ji} = f_i \text{ for every } i \leq j\}.$$

Similary, for an inverse system $\{(X_i), (\varphi_{ij} : X_j \to X_i)\}$, its inverse limit is the unique object representing the following contravariant functor $\mathcal{C} \to \mathbf{Sets}$:

$$Y \mapsto \{(g_i : Y \to X_i) : \varphi_{ij} \circ g_j = g_i \text{ for every } i \leq j\}.$$

Last time we saw some examples of the direct limit. Today, we will see another such example where the *right directedness* hypothesis is imposed on the poset $(I, \leq)$. Then, we will go over some examples of the inverse limit.

(7.1) Example involving right directed posets.– Assume that $(I, \leq)$ is a right directed poset. Recall that this means that for every $i, j \in I$, there exists $k \in I$ such that $i \leq k$ and $j \leq k$.

Let **CommRings** be the category of unital, commutative rings, where the morphisms are assumed to be unital as well. Let $\{(A_i)_{i\in I}, (\psi_{ji} : A_i \to A_j)_{i \leq j}\}$ be a direct system, valued in **CommRings**.

Theorem. *The direct limit* $A = \varinjlim_{(I,\leq)} A_i$ *exists in the category* **CommRings**. *Moreover, if each* A_i *is an integral domain* [1] *(resp. a field) then so is* A.

Compare this result with Example V of §5.4. The statement would be obviously false if the right directedness is not assumed, for instance, for a discrete poset (i.e, any two distinct elements are non-comparable).

Proof. We will construct A, as a set, by imposing an equivalence relation on the disjoint union of $A_i's$. Namely,

$$A := \left(\bigsqcup_{i \in I} A_i \right) / \sim \text{ , where}$$

$a \in A_i$ and $b \in A_j$ are *equivalent* iff there exists k, $i \leq k$ and $j \leq k$, such that $\psi_{ki}(a) = \psi_{kj}(b)$ in A_k. I am going to leave it for you to check that this is indeed an equivalence relation.

[1] Recall that a commutative ring A is an integral domain if $ab = 0$ implies either $a = 0$ or $b = 0$.

We first have to give A a structure of a ring, then check that it is unital and commutative. The canonical maps $A_i \to A$ are the natural inclusion followed by the natural surjection, i.e, $a \in A_i$ goes to the equivalence class containing a. We will have to make sure that these are (unital) ring homomorphisms, and that they satisfy the universal property of direct limits.

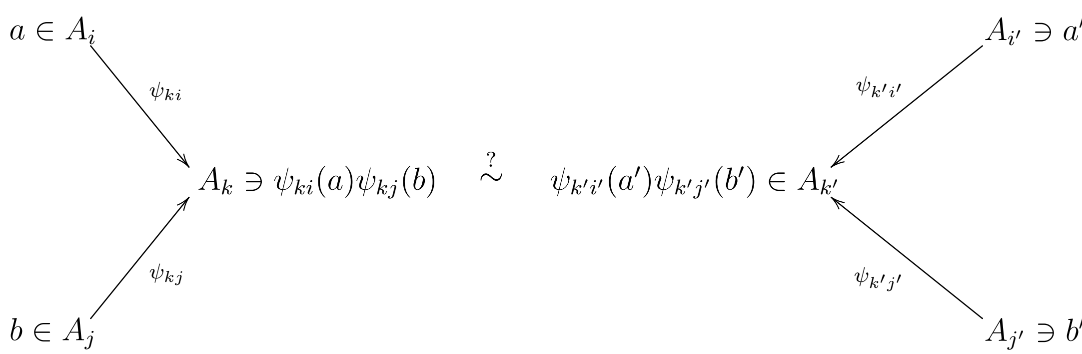
So, given two elements $\alpha, \beta \in A$, choose representatives $a \in \alpha$, $b \in \beta$. Let $i, j \in I$ be such that $a \in A_i$ and $b \in A_j$. Pick $k \in I$ such that $i \leq k$ and $j \leq k$, and define the sum and product as follows:

$$\alpha + \beta := \text{Equivalence class of } \psi_{ki}(a) + \psi_{kj}(b).$$

$$\alpha \cdot \beta := \text{Equivalence class of } \psi_{ki}(a)\psi_{kj}(b) \in A_k.$$

We now prove that these operations are well–defined. Let us show it for the product (the proof for the sum is exactly the same - replace multiplication by addition in the following argument).

Assume that we pick some representatives other than a, b, say $a' \in A_{i'}$ and $b' \in A_{j'}$ from α and β respectively, and take $k' \geq i', j'$ to get an element $\psi_{k'i'}(a')\psi_{k'j'}(b')$ in $A_{k'}$. We have to show that this element belongs to the same equivalence class as $\psi_{ki}(a)\psi_{kj}(b)$. See the picture below:



Since $a \sim a'$, there exists an element $i'' \geq i, i'$ for which $\psi_{i''i}(a) = \psi_{i''i'}(a')$ (similarly for $b \sim b'$, we denote such an index by j''). Let us choose an index $\ell \in I$ larger than any index from the finite set $\{i'', j'', k, k'\}$ (exists by the right–directedness hypothesis). It suffices to prove the following:

Claim. $\psi_{\ell k}(\psi_{ki}(a)\psi_{kj}(b)) = \psi_{\ell k'}(\psi_{k'i'}(a')\psi_{k'j'}(b'))$.

Proof of the claim. We have the following computation, using the transitivity property of ψ morphisms, and the fact that these are homomorphisms of rings:

$$\begin{aligned}\psi_{\ell k}(\psi_{ki}(a)\psi_{kj}(b)) &= \psi_{\ell i}(a)\psi_{\ell j}(b) = \psi_{\ell i''}(\psi_{i''i}(a))\psi_{\ell j''}(\psi_{j''j}(b))\\ &= \psi_{\ell i''}(\psi_{i''i'}(a'))\psi_{\ell j''}(\psi_{j''j'}(b')) = \psi_{\ell i'}(a')\psi_{\ell j'}(b')\\ &= \psi_{\ell k'}(\psi_{k'i'}(a'))\psi_{\ell k'}(\psi_{k'j'}(b')) = \psi_{\ell k'}(\psi_{k'i'}(a')\psi_{k'j'}(b')).\end{aligned}$$

Thus the claim is established.

The role of $0 \in A$ is played by the equivalence class containing $0 \in A_i$ (for any $i \in I$), and that of $1 \in A$ by the class containing $1 \in A_i$ (again, for any $i \in I$). It remains to check that A is indeed a unital, commutative ring. The following observation follows immediately from the right–directedness assumption:

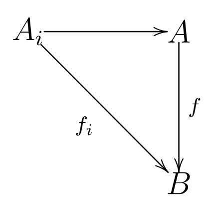
Observation. For a finite number of equivalence classes, $\{\alpha_1, \ldots, \alpha_n\}$, there exists $k \in I$ and representatives $a_1 \in \alpha_1, \ldots, a_n \in \alpha_n$ such that $a_1, \ldots, a_n \in A_k$.

Since axioms for A to be a unital, commutative ring only involve finitely many elements, and they hold for A_k, they will continue to hold for A. For instance, let us write down how the proof of associativity will proceed. Let α, β, γ be three elements of A. Pick representatives a, b, c from these equivalence classes, which all belong to the same A_k (for some $k \in I$). Since in A_k, $(ab)c = a(bc)$, it follows that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

Same argument as above goes into showing that, if each A_i is an integral domain (resp. a field), then so is A.

Finally, let us assume that $B \in$ **CommRings**, and we are given morphisms $f_i : A_i \to B$, for every $i \in I$, satisfying $f_j \circ \psi_{ji} = f_i$ for every $i \leq j$. Consider the unique (set) map we get $\widetilde{f} : \bigsqcup_{i \in I} A_i \to B$, whose restriction to A_i is f_i.

If $a \in A_i$ and $b \in A_j$ are equivalent, i.e, there exists $k \geq i, j$ such that $\psi_{ki}(a) = \psi_{kj}(b)$, then $f_i(a) = f_k(\psi_{ki}(a)) = f_k(\psi_{kj}(b)) = f_j(b)$. Hence, $\widetilde{f}$ factors through the equivalence relation to give $f : A \to B$. It is obvious (left as an easy exercise) to see that f is a ring homomorphism, and is the unique one which makes the following diagram commute, for every $i \leq j$:



where, recall that $A_i \to A$ is given by the following composition $A_i \subset \bigsqcup_{j \in I} A_j \twoheadrightarrow A$. □

(7.2) Example of inverse limit of groups.– Now, let us focus on inverse limits. As direct limits are contructed, in practice, as "quotients" of direct sums, inverse limits will be constructed as "subobjects" of direct product. The following lemma illustrates this point. We are **not** assuming any directedness hypothesis here.

Lemma. *Inverse limits exist in the category of groups.*

Proof. Let $(I, \leq)$ be a poset, and consider an inverse system of groups:

$$\{(G_i)_{i \in I}; (\varphi_{ij} : G_j \to G_i)_{i \leq j}\}$$

Recall that this means: (i) $\varphi_{ii} = \mathrm{Id}_{G_i}$ $(\forall i \in I)$, and (ii) $\varphi_{ij}\varphi_{jk} = \varphi_{ik}$ $(\forall i \leq j \leq k)$.

Consider $\widetilde{G} = \prod_{i\in I} G_i$ together with canonical projections $\widetilde{p_i} : \widetilde{G} \to G_i$. Let G be the following subset of $\widetilde{G}$:

$$G = \{(x_i \in G_i)_{i\in I} : \varphi_{ij}(x_j) = x_i \text{ for every } i \le j\}.$$

Note that G is a subgroup of $\widetilde{G}$. This is because, if $e_i \in G_i$ is the neutral element, then $e = (e_i)_{i\in I}$ is in G which is neutral with respect to componentwise multiplication. Moreover if $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ are two elements of G, then $(x_i^{-1}y_i)_{i\in I}$ is also in G, because:

$$\text{for every } i \le j,\ \varphi_{ij}(x_j^{-1}y_j) = \varphi_{ij}(x_j)^{-1}\varphi_{ij}(y_j) = x_i^{-1}y_i.$$

Now let $p_i : G \to G_i$ be the restriction of $\widetilde{p_i}$ to $G \subset \widetilde{G}$. We have: $\varphi_{ij} \circ p_j = p_i$, for every $i \le j$, because, once evaluated on a tuple $(x_k)_{k\in I}$, this becomes $\varphi_{ij}(x_j) = x_i$. This holds by definition of G.

Finally, let H be another group, and let $g_i : H \to G_i$ be group homomorphisms, for which $\varphi_{ij} \circ g_j = g_i$. Thus we get a unique group homomorphism $\widetilde{g} : H \to \widetilde{G}$, given by $x \mapsto (g_i(x))_{i\in I}$. All we need to check is that $\mathrm{Im}(g) \subset G$. That is, for every $i \le j$, $\varphi_{ij}(g_j(x)) = g_i(x)$. As it was assumed for the group homomorphisms $(g_i)_{i\in I}$, we are done. □

Remark. The same proof goes over to prove that inverse limits exist in the category of rings, abelian groups, modules over a ring, vector spaces and so on.

(7.3) A concrete example.– Consider $I = \mathbb{N}$ with its natural (total) order. For each $n \in \mathbb{N}$, let $A_n = \mathbb{C}[x]/(x^{n+1})$, an object of the category **CommRings**. For $m < n$, we take the following ring homomorphism:

$$\varphi_{mn} : \mathbb{C}[x]/(x^{n+1}) \to \mathbb{C}[x]/(x^{m+1}), \qquad \sum_{i=0}^{n} a_i x^i \mapsto \sum_{j=0}^{m} a_j x^j.$$

I am going to leave it as an exercise for you to prove the following (the limit is taken in the category **CommRings**):

$$\varprojlim_{n\in\mathbb{N}} \mathbb{C}[x]/(x^{n+1}) = \mathbb{C}[\![x]\!],$$

the ring of formal power series in a variable x.

(7.4) Completion of a ring with respect to an ideal.– Another application of inverse limits is to define completion of a ring with respect to an ideal. This generalizes the example presented in the previous paragraph, where the ring under consideration was $\mathbb{C}[x]$ and the ideal was $(x) \subset \mathbb{C}[x]$.

So, let $A \in$ **CommRings**, and let $\mathfrak{a} \subset A$ be an ideal. Recall that this means $\mathfrak{a}$ is an abelian subgroup of A (under addition), and for every $x \in A$, $y \in \mathfrak{a}$, we have $xy \in \mathfrak{a}$.

Consider the following inverse system in **CommRings**, with respect to $I = \mathbb{N}$.

$$A_n = A/\mathfrak{a}^{n+1}, \qquad \pi_{mn} : A_n \to A_m,\ \forall\ m \le n.$$

Here, π_{mn} is the natural projection.

The completion of A with respect to the ideal $\mathfrak{a}$ is defined as the inverse limit of this inverse system of rings.

$$\widehat{A} := \varprojlim_{n\in\mathbb{N}} A/\mathfrak{a}^{n+1}.$$

Note that there are natural ring homomorphisms $A \to A_n$, for every $n \in \mathbb{N}$, which give rise to a ring homomorphism (by the universal property of inverse limits) $A \to \widehat{A}$. For instance, in the set up of the previous paragraph, this is just the usual inclusion of polynomials into power series $\mathbb{C}[x] \subset \mathbb{C}[\![x]\!]$.

For instance, let us take $A = \mathbb{Z}$ and $\mathfrak{a} = (p)$, for a prime number $p \in \mathbb{Z}_{\geq 2}$. The resulting inverse limit is denoted by $\widehat{\mathbb{Z}}_p$, and is called *the ring of p–adic integers*:

$$\widehat{\mathbb{Z}}_p := \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/(p^{n+1}).$$