LECTURE 8

(8.0) Next topic.– Recall that so far we have been learning the language of category theory. We saw the meaning of the following terms: categories, functors, natural transformations, equivalence of categories, adjoint functors, representable functors (after Yoneda's embedding theorem), direct sums/products, direct and inverse limits.

Now we are going to focus on categories of more algebraic origin: *additive and abelian categories.*

(8.1) Additive categories.– An *additive category* $\mathcal{A}$ is a category satisfying the following axioms:

(A1) For every $X, Y \in \mathcal{A}$, the set of morphisms $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ has a structure of an abelian group, such that the composition map is $\mathbb{Z}$–bilinear. That is, for every $X, Y, Z \in \mathcal{A}$:

- $g \circ (a_1 f_1 + a_2 f_2) = a_1(g \circ f_1) + a_2(g \circ f_2)$ for every $a_1, a_2 \in \mathbb{Z}$, $f_1, f_2 \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{A}}(Y, Z)$.
- $(a_1 g_1 + a_2 g_2) \circ f = a_1(g_1 \circ f) + a_2(g_2 \circ f)$, for every $a_1, a_2 \in \mathbb{Z}$, $f \in \mathrm{Hom}_{\mathcal{A}}(X, Y)$ and $g_1, g_2 \in \mathrm{Hom}_{\mathcal{A}}(Y, Z)$.

(A2) There exists an object in $\mathcal{A}$, denoted by $\mathbf{0}_{\mathcal{A}}$, such that $\mathrm{Hom}_{\mathcal{A}}(\mathbf{0}_{\mathcal{A}}, X) = \{0\} = \mathrm{Hom}_{\mathcal{A}}(X, \mathbf{0}_{\mathcal{A}})$, for every $X \in \mathcal{A}$.

(A3) Finite direct sums and products exist in $\mathcal{A}$.

Remarks.

(1) You have seen in the recitations that, in any category $\mathcal{C}$, if there is an object T such that $\mathrm{Hom}_{\mathcal{C}}(T, X)$ and $\mathrm{Hom}_{\mathcal{C}}(X, T)$ are singletons, for every $X \in \mathcal{C}$, then this object T is unique up to a unique isomorphism. Thus $\mathbf{0}_{\mathcal{A}}$ in the axiom (A2) of an additive category is uniquely defined, called *the* trivial (or zero) object of $\mathcal{A}$.

(2) In an additive category $\mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ is not empty, for any $X, Y \in \mathcal{A}$. This is because it is supposed to be an abelian group and as such must contain the zero morphism. Hence, for instance, the category of fields is not additive.

(3) In (A3), the finite direct sums and products that are assumed to exist in $\mathcal{A}$ end up being naturally isomorphic. See Lemma (8.3) below.

(8.2) Examples.– Many categories outside of the realm of algebra are not additive. For instance, category of topological spaces, manifolds etc. Categories of groups, unital rings, fields are also not additive.

Most typical example of an additive category is that of left (resp. right) modules over a unital ring R, which we denoted by R-mod (resp. mod-R). Let us see this in detail.

Recall that an object of R-mod is an abelian group M, together with a left action of R, which can be written as a (unital) ring homomorphism $\lambda_M : R \to \mathrm{End}_{\mathbb{Z}}(M)$. We often suppress this λ_M from our notations, and simply write $rm = \lambda_M(r)(m)$.

$$\mathrm{Hom}_{R\text{-mod}}(M,N) = \{f \in \mathrm{Hom}_{\mathbf{Ab}}(M,N) : f(rm) = rf(m)\ \forall\ r \in R, m \in M\}.$$

(A1): For $M, N \in R$-mod, the set of morphisms $\mathrm{Hom}_R(M,N)$ has a structure of an abelian group, given explicitly as follows.

- For $f_1, f_2 \in \mathrm{Hom}_R(M,N)$, $f_1 + f_2 \in \mathrm{Hom}_R(M,N)$ is defined as $(f_1+f_2)(m) = f_1(m) + f_2(m)$, for every $m \in M$.
- For $f : M \to N$, $(-f) : M \to N$ is defined as $(-f)(m) = -f(m)$, for every $m \in M$.
- Zero morphism is the one that maps $m \mapsto 0$, for every $m \in M$.

It is easy to see that the composition is bilinear.

(A2): The zero object is the trivial abelian group $\{0\}$ together with the trivial action of R.

(A3): Given $M_1, M_2 \in R$-mod, their direct sum is constructed as a cartesian product, with componentwise R–action: $M_1 \oplus M_2 = \{(m_1, m_2) : m_j \in M_j\ j = 1, 2\}$. For every $r \in R$, $m_1 \in M_1$ and $m_2 \in M_2$, $r(m_1, m_2) = (rm_1, rm_2)$.

(8.3) Direct sums and products.– Let I be an indexing set, $\mathcal{A}$ an additive category and $\{X_i\}_{i\in I}$ a set of objects from $\mathcal{A}$. Assume that both their direct sum $X = \bigoplus_{i\in I} X_i$ and direct product $\widehat{X} = \prod_{i\in I} X_i$ exist in $\mathcal{A}$. Let us denote the canonical morphisms as:

$$X_i \xrightarrow{f_i} X, \qquad \widehat{X} \xrightarrow{f^i} X_i,\ \forall i \in I.$$

For $i, j \in I$, define a morphism $\delta_{ij} : X_j \to X_i$ as: $\delta_{ij} = \begin{cases} 0 & i \neq j, \\ \mathrm{Id}_{X_i} & i = j. \end{cases}$

Lemma. *There is a natural morphism $\alpha : X \to \widehat{X}$ which makes the following diagram commute, for every $i, j \in I$.*

$$\begin{tikzcd} X_i \arrow[r, "\delta_{ji}"] \arrow[d, "f_i"'] & X_j \\ X \arrow[r, "\alpha"'] & \widehat{X} \arrow[u, "f^i"'] \end{tikzcd}$$

Moreover, if I is finite, α is an isomorphism.

Proof. The morphism α can be obtained by using the universal property of direct sums and products. Namely, fix $i \in I$, and consider the collection of morphisms $\{X_i \xrightarrow{\delta_{ji}} X_j\}_{j\in I}$. By the universal property of direct products, we get a unique morphism $g_i : X_i \to \widehat{X}$ such that $f^j \circ g_i = \delta_{ji}$ for every $j \in I$.

Now take the set of morphisms $\{X_i \xrightarrow{g_i} \widehat{X}\}_{i\in I}$. By the universal property of direct sums, we get a unique morphism $\alpha : X \to \widehat{X}$ such that $\alpha \circ f_i = g_i$. Combining this equation with the last one the previous paragraph, we get:

$$f^j \circ \alpha \circ f_i = f^j \circ g_i = \delta_{ji},$$

as claimed in the commutative diagram above.

I am going to give naturality of α as a homework problem. Namely, to show that $\alpha : \bigoplus_{i\in I} \Rightarrow \prod_{i\in I}$ is a natural transformation of functor $\mathcal{A}^I \to \mathcal{A}$ (assuming direct sums and products over I exist).

Let us see that when $|I| < \infty$, α is an isomorphism. Let $\beta : \widehat{X} \to X$ be given by:

$$\beta = \sum_{i\in I} f_i \circ f^i \in \mathrm{Hom}_{\mathcal{A}}(\widehat{X}, X).$$

Note that the sum is taken in the indicated abelian group, and makes sense, since I is finite.

$\beta \circ \alpha = \mathrm{Id}_X$: Note that, by considering the collection of morphisms $\{f_i : X_i \to X\}_{i\in I}$, and the universal property of direct sums, we know there to be a unique morphism $h : X \to X$ for which $h \circ f_i = f_i$. Hence, $h = \mathrm{Id}_X$. Therefore, it suffices to show that $(\beta \circ \alpha) \circ f_i = f_i$ for every $i \in I$.

$$\beta \circ \alpha \circ f_i = \sum_{j\in I} f_j \circ (f^j \circ \alpha \circ f_i) = \sum_{j\in I} f_j \circ \delta_{ji} = f_i.$$

Here, we have used that $- \circ f_i$ distributes over addition of morphisms, associativity of composition, and that $f^j \circ \alpha \circ f_i = \delta_{ji}$ as shown above.

The proof of $\alpha \circ \beta = \mathrm{Id}_{\widehat{X}}$ is similar, and is omitted here. □

(8.4) Kernel and cokernel of a morphism.– Recall from §0.4, that a morphism $f : X \to Y$ in an arbitrary category $\mathcal{C}$ is said to be injective (resp. surjective) if it can be cancelled from the left (resp. right). That is $f \circ h_1 = f \circ h_2$ implies $h_1 = h_2$ (resp. $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$).

For an additive category $\mathcal{A}$, the definition above can be rewritten as follows. $f : X \to Y$ in $\mathcal{A}$ is injective (resp. surjective) if and only if $f \circ h = 0$ implies $h = 0$ (resp. $g \circ f = 0$ implies $g = 0$).

Definition. Let $\mathcal{A}$ be an additive category, and let $f : X \to Y$ be a morphism in $\mathcal{A}$. *Kernel* of f is a pair (K, i), where $K \in \mathcal{A}$ and $i : K \to X$, satisfying:

- $f \circ i = 0$, and
- for any morphism $g : Z \to X$, such that $f \circ g = 0$, there is a unique morphism $\overline{g} : Z \to K$ such that $g = i \circ \overline{g}$.

Similarly, *Cokernel* of f is a pair (C, π), where $C \in \mathcal{A}$ and $\pi : Y \to C$, satisfying:

- $\pi \circ f = 0$, and
- for any morphism $h : Y \to Z$, such that $h \circ f = 0$, there is a unique morphism $\overline{h} : C \to Z$ such that $h = \overline{h} \circ \pi$.

If exist, kernel and cokernel of a morphism are unique, up to unique isomorphism. This is easily seen by rephrasing the definition given above, as representability of a functor. Namely, given $f : X \to Y$ as before, consider the following two functors (contravariant and covariant respectively):

$$\mathcal{K}(f) : \mathcal{A} \to \mathbf{Ab}, \qquad \mathcal{P}(f) : \mathcal{A} \to \mathbf{Ab}.$$

where, on objects:

$$\mathcal{K}(f) : Z \mapsto \left\{ Z \xrightarrow{g} X : f \circ g = 0 \right\}, \qquad \text{and} \qquad \mathcal{P}(f) : Z \mapsto \left\{ Y \xrightarrow{h} Z : h \circ f = 0 \right\}.$$

On morphisms, these functors act exactly as the contravariant and covariant hom functors h^X, h_Y. Namely, $\mathcal{K}(f) : a \mapsto - \circ a$, and $\mathcal{P}(f) : b \mapsto b \circ -$.

Exercise. Verify that kernel and cokernel of f (if exist) represent $\mathcal{K}(f)$ and $\mathcal{P}(f)$ respectively.

This exercise implies the claimed uniqueness. The kernel of f (resp. cokernel of f) is denoted as usual by $\mathrm{Ker}(f)$ (resp. $\mathrm{CoKer}(f)$), often suppressing the morphism i (resp. π) from the notation.

(8.5) Properties of kernels and cokernels.– The definition given above is often depicted pictorially as follows. Again, $\mathcal{A}$ is an additive category and $f : X \to Y$ is morphism. $\mathrm{Ker}(f) \xrightarrow{i} X$ is the unique object fitting in the following diagram:

$$\begin{tikzcd}
\boxed{\mathrm{Ker}(f)} \arrow[r, "i"] \arrow[rr, bend left, "=0"] & X \arrow[r, "f"] & Y \\
& Z \arrow[ul, dashed, "\exists!\ \overline{g}"] \arrow[u, "\forall g"'] \arrow[ur, "=0"'] &
\end{tikzcd}$$

Similarly, $\mathrm{CoKer}(f)$ fits in the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[rr, bend left, "=0"] \arrow[dr, "=0"'] & Y \arrow[r, "\pi"] \arrow[d, "\forall h"'] & \boxed{\mathrm{CoKer}(f)} \arrow[dl, dashed, "\exists!\ \overline{h}"] \\
 & Z &
\end{tikzcd}$$

Proposition. *Let $\mathcal{A}$ be an additive category, $f : X \to Y$ a morphism in $\mathcal{A}$, and assume that its kernel and cokernel exist.*

(1) *The morphism $i : \mathrm{Ker}(f) \to X$ is injective. Similarly, the morphism $\pi : Y \to \mathrm{CoKer}(f)$ is surjective.*

(2) *f is injective if, and only if $(\mathrm{Ker}(f), i)$ is naturally isomorphic to $(\mathbf{0}_{\mathcal{A}}, 0)$. Here $0 \in \mathrm{Hom}_{\mathcal{A}}(\mathbf{0}_{\mathcal{A}}, X)$. Similarly, f is surjective if, and only if $(\mathrm{CoKer}(f), \pi)$ is naturally isomorphic to $(\mathbf{0}_{\mathcal{A}}, 0)$.*

Proof. Let us prove the two assertions for kernel only. The proof for cokernel is similar.

Proof of (1): To show that i is injective, we need to prove that it can be cancelled from the left. That is, $i \circ a = 0$ implies $a = 0$ (here, $a : Z \to \mathrm{Ker}(f)$ is an arbitrary morphism in $\mathcal{A}$). Take $g = 0 : Z \to X$ so that $f \circ g = 0$. Now both a and 0 work as $\overline{g}$, hence by its uniqueness, must be equal.

Proof of (2): Assume that f is injective. Recall that this means $f \circ a = 0$ implies $a = 0$. We will prove that $(\mathbf{0}_{\mathcal{A}}, 0)$ satisfies the universal property for being kernel of f. It is clear that $f \circ 0 = 0$. So, let us assume that we are given $g : Z \to X$ such that $f \circ g = 0$. By injectivity of f, we get that $g = 0$, and hence factors uniquely through $Z \xrightarrow{0} \mathbf{0}_{\mathcal{A}} \xrightarrow{0} X$.

Conversely, assuming $0 : \mathbf{0}_{\mathcal{A}} \to X$ is the kernel of f, we will prove that f is injective. Again let $g : Z \to X$ be such that $f \circ g = 0$, and we need to show $g = 0$. By the universal property of kernel, g must factor through $\mathbf{0}_{\mathcal{A}}$, that is, $g = 0 \circ \overline{g} = 0$. $\square$