LECTURE 9

(9.0) Additive categories.– Recall that in the last lecture we defined additive categories. An additive category $\mathcal{A}$ is a category where: (A1) Hom sets have a structure of an abelian group such that the composition map is $\mathbb{Z}$–bilinear, (A2) trivial object $\mathbf{0}_{\mathcal{A}}$ exists, and (A3) finite direct sums and products exist.

We proved that finite direct sums and products end up being isomorphic naturally. We defined kernel and cokernel of a morphism $f : X \to Y$ in $\mathcal{A}$. Let us copy the diagrams showing the universal property of kernels and cokernels from the last lecture:

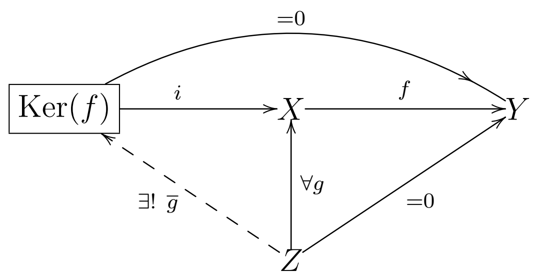


FIGURE 1. Kernel of a morphism

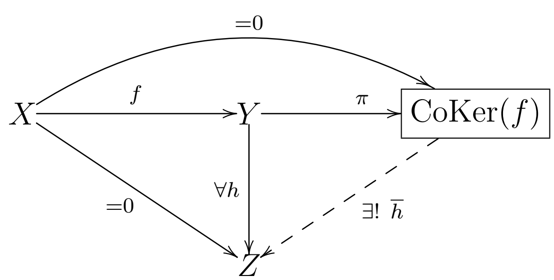


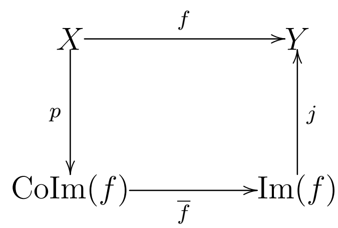
FIGURE 2. Cokernel of a morphism

(9.1) Image and coimage of a morphism.– Let $f : X \to Y$ be a morphism in an additive category $\mathcal{A}$.

Definition. The *image of* f, denoted by $\mathrm{Im}(f)$, is defined as *the kernel of the cokernel*. Namely, if $(C, \pi : Y \to C)$ is the cokernel of f, then $\mathrm{Im}(f)$ is defined to be the kernel of π.

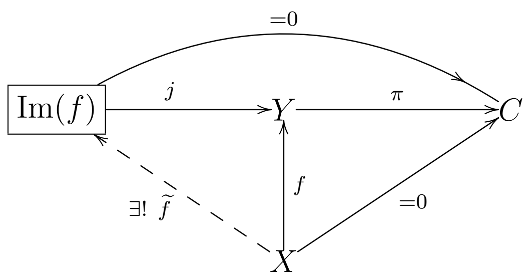
Similarly, *coimage of* f, denoted by $\mathrm{CoIm}(f)$, is *the cokernel of the kernel*. That is, if $(K, i : K \to X)$ is the kernel of f, then $\mathrm{CoIm}(f)$ is the kernel of i. We will unfold these definitions in the proof of the lemma below.

Lemma. *Let $f : X \to Y$ be a morphism in an additive category $\mathcal{A}$. Assuming that the relevant kernels and cokernels exist, we have a commutative diagram:*

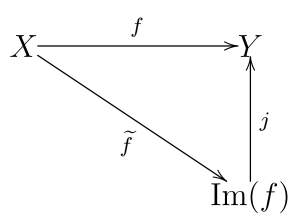


The morphism j is injective and p is surjective.

Proof. Let $(K, i : K \to X)$ be the kernel and $(C, \pi : Y \to C)$ be the cokernel of f. Now $\mathrm{Im}(f)$, being the kernel of π fits in a diagram similar to the one in Figure 1. We draw it for the test morphism $f : X \to Y$ for which we know that $\pi \circ f = 0$.



Thus, we conclude that there is a unique morphism $\widetilde{f} : X \to \mathrm{Im}(f)$ such that $f = j \circ \widetilde{f}$. Moreover, j being the canonical morphism for a kernel is injective (see Prop. 8.5 (i)).

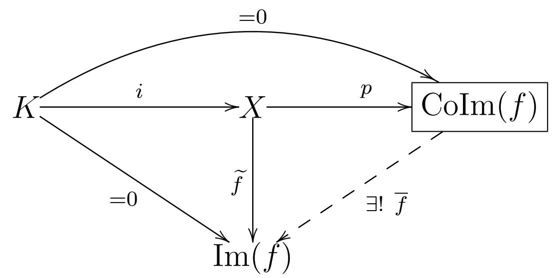


Claim. $\mathrm{Ker}(f) = \mathrm{Ker}(\widetilde{f})$.

Proof of the claim. This is true in general that composing a morphism, on the left, by an injective morphism does not change its kernel. We will show that $(K, i : K \to X)$ (kernel of f) satisfies the universal property of the $\mathrm{Ker}(\widetilde{f})$.

- $\widetilde{f} \circ i = 0$. This is because $j \circ (\widetilde{f} \circ i) = (j \circ \widetilde{f}) \circ i = f \circ i = 0$, and j is injective.
- If there is a morphism $g : Z \to X$ such that $\widetilde{f} \circ g = 0$, then $g = i \circ \overline{g}$ for a unique $\overline{g} : Z \to K$. Again, this is because $0 = j \circ (\widetilde{f} \circ g) = f \circ g$ and the assertion follows by the definition of $\mathrm{Ker}(f)$.

Now we focus on the coimage of $\widetilde{f}$, which is defined as the cokernel of the kernel $\mathrm{Ker}(\widetilde{f}) = \mathrm{Ker}(f) \xrightarrow{i} X$. Thus it follows that $\mathrm{CoIm}(f) = \mathrm{CoIm}\,\widetilde{f}$. We fit it in a defining picture, akin to the one given in Figure 2, applied to $\widetilde{f}$.



Again, p is surjective, being the canonical homomorphism of a cokernel (Prop. 8.5 (i)). This completes the diagram stated in the lemma, and finished the proof. □

(9.2) Remarks.–

1. Note that we have not said anything about $\overline{f}$ being bijective in the statement of the lemma above. It is true assuming the kernels and cokernels of *all* morphisms exist in $\mathcal{A}$. I am not going to prove it here, since it will just break the flow and introduce unnecessary complications.

2. If f was bijective to begin with, then $\mathrm{Ker}(f)$ and $\mathrm{CoKer}(f)$ are both $0_{\mathcal{A}}$ (see Prop. 8.5 (ii)), and hence $\mathrm{CoIm}(f) = X$ and $\mathrm{Im}(f) = Y$, so $f = \overline{f}$.

(9.3) Additive functors.– Let $\mathcal{A}$ and $\mathcal{B}$ be two additive categories, and $F : \mathcal{A} \to \mathcal{B}$ be a functor. We say that F is an *additive functor* if for every $A_1, A_2 \in \mathcal{A}$, the resulting map $F : \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \to \mathrm{Hom}_{\mathcal{B}}(F(A_1), F(A_2))$ is a group homomorphism.

In the context of additive (or abelian, as defined below) categories, we only speak of additive functors, and often forget the adjective *additive*.

Example. Note that for an additive category $\mathcal{A}$, the contravariant and covariant hom functors $\mathsf{h}^X, \mathsf{h}_X$ take values in the category of abelian groups.

$$\mathsf{h}^X, \mathsf{h}_X : \mathcal{A} \to \mathbf{Ab}.$$

The axiom of composition being bilinear (i.e, it distributes over addition/subtraction of morphisms) is same as saying that these functors are additive.

(9.4) Abelian categories.– An additive category $\mathcal{A}$ is *abelian* if the following two conditions hold (i.e, in addition to the axioms (A1)–(A3) of additive categories - see §8.1):

(AB1) Kernels and cokernels exist for every morphism in $\mathcal{A}$.

(AB2) For every morphism $f : X \to Y$, the morphism $\overline{f} : \mathrm{CoIm}(f) \to \mathrm{Im}(f)$ is an *isomorphism.*

Thus the *first isomorphism theorem* for abelian groups, or more generally R–modules, is an *axiom* for abelian categories. Note that, if f is a bijection, then $f = \overline{f}$ is always an isomorphism in abelian categories.

Recall that in the category of filtered abelian groups (see §0.7) there are bijections which are not isomorphisms. This is one of the standard examples of an additive category which is not abelian.

Example. Let R be a unital (not necessarily commutative) ring, and let $\mathcal{A} = R$-mod be the category of left R–modules. We showed (in §8.2) that this category is additive. This is also a prototypical example of an abelian category:

Axiom (AB1). Given an R–linear map $f : M \to N$, between two (left) R–modules, M and N, we define its kernel and cokernel as usual:

$$\operatorname{Ker}(f) := \{m \in M : f(m) = 0\} \subset M, \qquad N \twoheadrightarrow \operatorname{CoKer}(f) := N/\{f(m) : m \in M\}.$$

It is easy to verify that these satisfy their respective universal properties. Moreover,

$$M \twoheadrightarrow \operatorname{CoIm}(f) = M/\operatorname{Ker}(f), \qquad \operatorname{Im}(f) = \{f(m) : m \in M\} \subset N.$$

Axiom (AB2). Verifying this axiom is same as proving the first isomorphism theorem for R–modules. Namely,

$$\overline{f} : M/\operatorname{Ker}(f) \to \operatorname{Im}(f), \qquad m + \operatorname{Ker}(f) \mapsto f(m),$$

is an isomorphism. That this map is well–defined is the easy check omitted above. Let us prove that it is invertible, hence an isomorphism. Define $g : \operatorname{Im}(f) \to M/\operatorname{Ker}(f)$ as follows. For $x \in \operatorname{Im}(f)$, choose $m \in M$ such that $f(m) = x$. Then $g(x) := m + \operatorname{Ker}(f)$.

Well–defined. If $x = f(m) = f(m')$, then $m - m' \in \operatorname{Ker}(f)$, hence $g(x) = m + \operatorname{Ker}(f) = m' + \operatorname{Ker}(f)$.

g is R–linear. Left as an easy exercise.

Now, for $x = f(m) \in \operatorname{Im}(f)$, $f(g(x)) = f(m + \operatorname{Ker}(f)) = f(m) = x$. So, $f \circ g = \operatorname{Id}_{\operatorname{Im}(f)}$. Similarly for $m + \operatorname{Ker}(f) \in M/\operatorname{Ker}(f)$, $g(f(m)) = m + \operatorname{Ker}(f)$ by definition, so $g \circ f = \operatorname{Id}_{M/\operatorname{Ker}(f)}$. Hence f and g are isomorphisms.

Remark. These axioms were formulated in Grothendieck's Tôhoku paper (1957) *Sur quelques points d'algèbre homologique.*

(9.5) Another family of examples.– Given an arbitrary category $\mathcal{C}$ and an additive (resp. abelian) category $\mathcal{A}$, the categories of functors $\mathcal{F}(\mathcal{C}, \mathcal{A})$ and $\mathcal{F}(\mathcal{C}^{\mathrm{op}}, \mathcal{A})$ get a structure of an additve (resp. abelian) category. This assertion is analoguous to the fact that set of maps to an algebraic structure, inhert that structure by pointwise operations.

For instance, if $F, G \in \mathcal{F}(\mathcal{C}, \mathcal{A})$ are two functors, and $\xi, \eta : F \Rightarrow G$ are two natural transformations, then, $\xi + \eta : F \Rightarrow G$ is given by $(\xi + \eta)_X = \xi_X + \eta_X$, where the addition is carried out in $\operatorname{Hom}_{\mathcal{A}}(F(X), G(X))$. Similarly $(-\xi)_X = -\xi_X$ for every $X \in \mathcal{C}$. Since composition of natural transformations is also pointwise, it is easy to see that it is bilinear with respect to the given abelian group structure on the set of natural transformations, i.e, $\mathrm{Hom}_{\mathcal{F}(\mathcal{C},\mathcal{A})}(F,G)$.

The role of *trivial functor* is played by $\mathbf{0} : \mathcal{C} \to \mathcal{A}$ which sends every object to $\mathbf{0}_{\mathcal{A}}$ and every morphism to the only morphism in $\mathrm{End}_{\mathcal{A}}(\mathbf{0}_{\mathcal{A}})$.

Direct sum of two functors $F, G : \mathcal{C} \to \mathcal{A}$ is also defined pointwise: $(F \oplus G)(X) = F(X) \oplus G(X)$. Thus, if $\mathcal{A}$ is additive, so is $\mathcal{F}(\mathcal{C},\mathcal{A})$.

Exercise. Assume that $\mathcal{A}$ is abelian. Construct *kernel* (resp. *cokernel*) of $\eta : F \Rightarrow G$ and verify that it satisfies the respective universal property. (*Hint:* $\mathrm{Ker}(\eta) : \mathcal{C} \to \mathcal{A}$ *sends* $X \in \mathcal{C}$ *to* $\mathrm{Ker}(\eta_X : F(X) \to G(X)) \in \mathcal{A}$).

Proposition. *Axiom (AB2) holds in* $\mathcal{F}(\mathcal{C},\mathcal{A})$. *Hence,* $\mathcal{F}(\mathcal{C},\mathcal{A})$ *is an abelian category.*

Proof. Let $\eta : F \Rightarrow G$ be a morphism in the functor category $\mathcal{F}(\mathcal{C},\mathcal{A})$. By the exercise given above, we have:

- $\pi : F \Rightarrow \mathrm{CoIm}(\eta)$ sends each object $X \in \mathcal{C}$ to $\pi_X : F(X) \twoheadrightarrow \mathrm{CoIm}(\eta_X)$.
- $\iota : \mathrm{Im}(\eta) \Rightarrow G$ sends $X \in \mathcal{C}$ to $\iota_X : \mathrm{Im}(\eta_X) \hookrightarrow G(X)$.
- $\overline{\eta} : \mathrm{CoIm}(\eta) \to \mathrm{Im}(\eta)$ evaluated on $X \in \mathcal{C}$ is $(\overline{\eta})_X = \overline{\eta_X} : \mathrm{CoIm}(\eta_X) \to \mathrm{Im}(\eta_X)$.

Hence, for every $X \in \mathcal{C}$, $(\overline{\eta})_X = \overline{\eta_X}$ is an isomorphism in $\mathcal{A}$, by (AB2) for $\mathcal{A}$. We know that taking pointwise inverse $\mu_X = (\overline{\eta}_X)^{-1} : \mathrm{Im}(\eta_X) \to \mathrm{CoIm}(\eta_X)$ gives rise to another natural transformation, which is the inverse of $\overline{\eta}$. Therefore (AB2) holds. □