LECTURE 10

(10.0) Abelian categories and additive functors.– Recall that we defined abelian categories as categories satisfying five axioms (A1)–(A3) (for being additive), and (AB1) (AB2) (see §8.1 and §9.4). Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian (or additive) categories, and $F : \mathcal{A} \to \mathcal{B}$ be a functor. We say F *is additive* if for every $X, Y \in \mathcal{A}$, the following map is a group homomorphism:

$$\mathrm{Hom}_{\mathcal{A}}(X, Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{B}}(F(X), F(Y)).$$

As an example, the covariant and contravariant Hom functors are additive.

(10.1) Some properties of additive functors.– Given an additive functor $F : \mathcal{A} \to \mathcal{B}$ between two abelian categories, we have the following.

Proposition.

(1) $F(\mathbf{0}_{\mathcal{A}}) \cong \mathbf{0}_{\mathcal{B}}$.

(2) *For every* $X_1, X_2 \in \mathcal{A}$, $F(X_1 \oplus X_2) \cong F(X_1) \oplus F(X_2)$.

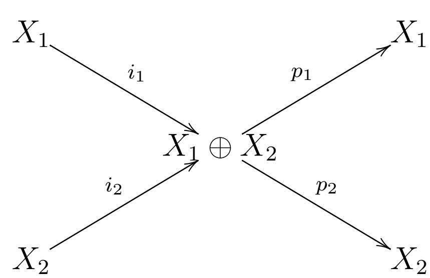
(3) *If* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *are two morphisms in* $\mathcal{A}$, *such that* $g \circ f = 0$, *then* $F(g) \circ F(f) = 0$.

Proof. (1). We remark that the trivial object of an additive category, say C of $\mathcal{C}$, is uniquely determined by the condition: $\mathrm{Id}_C = 0 \in \mathrm{End}_{\mathcal{C}}(C)$. This is clearly necessary, and to see sufficiency, we have that for any $D \in \mathcal{C}$ and $a : C \to D$, $a = a \circ \mathrm{Id}_C = a \circ 0 = 0$. Hence $\mathrm{Hom}_{\mathcal{C}}(C, D) = \{0\}$. Similarly, $\mathrm{Hom}_{\mathcal{C}}(D, C) = \{0\}$.

Now the proof of (1) is clear, since $F(\mathrm{Id}_{\mathbf{0}_{\mathcal{A}}}) = \mathrm{Id}_{F(\mathbf{0}_{\mathcal{A}})}$ by definition of a functor, and $F(0) = 0$ since it is a group homomorphism on the level of morphisms. Thus $\mathrm{Id}_{F(\mathbf{0}_{\mathcal{A}})} = F(0) = 0$ proving that $F(\mathbf{0}_{\mathcal{A}})$ is the trivial object of $\mathcal{D}$.

(3) is again easy, since $F(g) \circ F(f) = F(g \circ f) = F(0) = 0$.

Let us now prove (2). Let $i_\ell : X_\ell \to X_1 \oplus X_2$ and $p_\ell : X_1 \oplus X_2 \to X_\ell$ ($\ell = 1, 2$) be the canonical morphisms of direct sum, and product respectively (recall that the two are isomorphic by Lemma 8.3).



We have the following equations for these morphisms (see the proof of Lemma 8.3):

$$\boxed{p_k \circ i_\ell = \delta_{k\ell} \qquad \text{and} \qquad i_1p_1 + i_2p_2 = \mathrm{Id}_{X_1\oplus X_2}}$$

Here, recall that $\delta_{k\ell} = 0$ if $k \neq \ell$, and $= \mathrm{Id}_{X_\ell}$ if $k = \ell$. We claim that these equations determine the direct sum = direct product of two objects.

Claim. Let $X \in \mathcal{A}$, and assume that there are morphisms $f_\ell : X_\ell \to X$, $g_\ell : X \to X_\ell$ $(\ell = 1, 2)$ such that:

$$g_k \circ f_\ell = \delta_{k\ell}, \qquad f_1g_2 + f_2g_2 = \mathrm{Id}_X,$$

then $X \cong X_1 \oplus X_2$.

Given the claim, (3) follows since these equations clearly hold for $F(i_\ell) : F(X_1) \to F(X_1 \oplus X_2)$ and $F(p_\ell) : F(X_1 \oplus X_2) \to F(X_\ell)$, $(\ell = 1, 2)$.

Proof of the claim. We have to show that for any pair of morphisms $\{a_\ell : X_\ell \to Y\}_{\ell=1,2}$, there is a unique $a : X \to Y$ such that $a_\ell = a \circ f_\ell$. To see existence, set $a = a_1g_1 + a_2g_2$. Then, $a \circ f_\ell = \sum_{k=1,2} a_kg_kf_\ell = \sum a_k\delta_{k\ell} = a_\ell$.

To see uniqueness, if a and $\widetilde{a}$ are two such morphisms, then:

$$a = a \circ \mathrm{Id}_X = a(f_1g_1 + f_2g_2) = a_1g_1 + a_2g_2 = \widetilde{a}f_1g_1 + \widetilde{a}f_2g_2 = \widetilde{a} \circ \mathrm{Id}_X = \widetilde{a}.$$

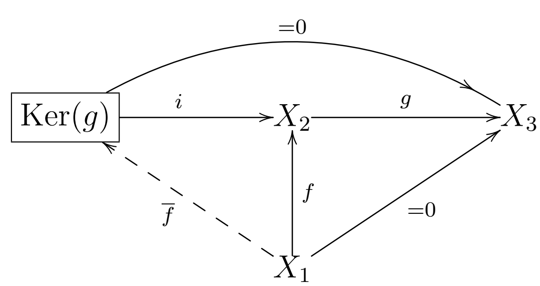
□

(10.2) Exact sequences.– Let $\mathcal{A}$ be an abelian category. Consider the following morphisms in $\mathcal{A}$.

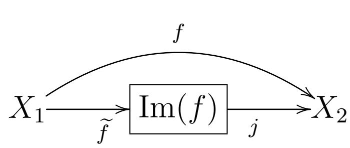
$$X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$$

We say that this sequence is *exact at* X_2 if $g \circ f = 0$ and $\mathrm{Ker}(g) = \mathrm{Im}(f)$.

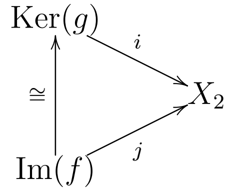
Remark. Note that requiring $g \circ f = 0$ is essential to sensibly ask for $\mathrm{Ker}(g)$ to be "same" as $\mathrm{Im}(f)$. Meaning, $g \circ f = 0$ implies, by the defining property of the kernel, that f factors through the kernel of g:



Similarly, as we saw in the proof of Lemma 9.1, f also factors through



Thus, strictly speaking we are asking for an isomorphism $\mathrm{Im}(f) \to \mathrm{Ker}(g)$ which makes the following diagram commute:



However, it has become a routine practice to write "$\mathrm{Im}(f) = \mathrm{Ker}(g)$" and we will continue to write it this way, the precise meaning assumed to be understood.

(10.3) Complexes, short exact sequences.– A *complex* valued in an abelian category $\mathcal{A}$, is a sequence of morphisms:

$$\cdots X_k \xrightarrow{f_k} X_{k+1} \xrightarrow{f_{k+1}} X_{k+2} \xrightarrow{f_{k+2}} \cdots$$

such that $f_{k+1} \circ f_k = 0$ for every $k \in \mathbb{Z}$. We say that this complex is *acyclic* if it is exact at X_k for every k, that is, $\mathrm{Im}(f_k) = \mathrm{Ker}(f_{k+1})$. We will use the terms *exact sequence* and *acyclic complex* to mean the same thing.

A *short exact sequence* is a sequence:

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \ ,$$

which is exact at X, Y and Z. In more detail:

- f is injective,
- g is surjective, and
- $\mathrm{Im}(f) = \mathrm{Ker}(g)$.

Note that a short exact sequence equivalently means that $(X, f : X \to Y)$ is the kernel of g and $(Y, g : Y \to Z)$ is the cokernel of f.

Example. Let R be an integral domain (i.e, R is a unital, commutative ring without zero–divisors), and $a \in R$. The following is a short exact sequence of R–modules:

$$0 \to R \xrightarrow{\mu_a} R \to R/(a) \to 0.$$

Here, $\mu_a : R \to R$ is multiplication by a, $(a) \subset R$ is the ideal generated by a, and $R \to R/(a)$ is the canonical surjection.

Example. Let $R = K[x, y]$ where K is a field. Consider the natural projection $\pi : R \to K \cong R[x, y]/(x, y)$. Let $R^2 = R \oplus R$, with elements $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Prove that the following is an exact sequence of R–modules:

$$0 \to R \xrightarrow{f} R^2 \xrightarrow{g} R \xrightarrow{\pi} K \to 0 \ ,$$

where,

- f sends $1 \mapsto ye_1 - xe_2$,
- g sends $e_1 \mapsto x$ and $e_2 \mapsto y$.

(10.4) Left and right exact functors.– Now let $F : \mathcal{A} \to \mathcal{B}$ be an additive, covariant functor (resp. $G : \mathcal{A} \to \mathcal{B}$ an additive contravariant functor). By Proposition 10.1 (3) above, F (and G) send complexes valued in $\mathcal{A}$ to those valued in $\mathcal{B}$.

Definition. We say F is *left exact*, if for every short exact sequence in $\mathcal{A}$:

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0,$$

the following is a short exact sequence in $\mathcal{B}$:

$$0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z).$$

F is said to be *right exact*, if for every short exact sequence in $\mathcal{A}$ as above, we have the following short exact sequence in $\mathcal{B}$:

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \to 0.$$

These notions for the contravariant functor G take the following form:

Left exactness: $0 \to G(Z) \xrightarrow{G(g)} G(Y) \xrightarrow{G(f)} G(X)$ is exact.

Right exactness: $G(Z) \xrightarrow{G(g)} G(Y) \xrightarrow{G(f)} G(X) \to 0$ is exact.

Functors that are both left and right exact are simply called *exact.*

Theorem. *Let $\mathcal{A}$ be an abelian category and $X \in \mathcal{A}$ an object. The contravariant and covariant* Hom *functors:*

$$\mathsf{h}^X, \mathsf{h}_X : \mathcal{A} \to \mathbf{Ab},$$

are both left exact.

Proof. Let us prove this theorem for the contravariant Hom functor h^X. Namely, we have to show that, if $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is a short exact sequence in $\mathcal{A}$, then the following is an exact sequence of abelian groups:

$$0 \to \mathrm{Hom}(C, X) \xrightarrow{-\circ g} \mathrm{Hom}(B, X) \xrightarrow{-\circ f} \mathrm{Hom}(A, X).$$

Step 1: $\mathsf{h}^X(g) = - \circ g$ is injective.

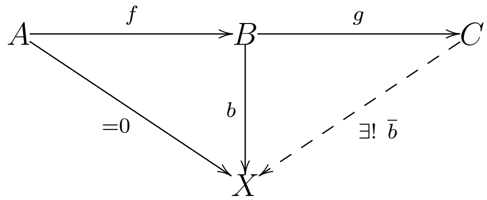
Proof. Let $c : C \to X$ be such that $\mathsf{h}^X(g)(c) = c \circ g = 0$. Since g is surjective, this means that $c = 0$, hence $\mathsf{h}^X(g)$ is injective.

Step 2: $\mathsf{h}^X(f) \circ \mathsf{h}^X(g) = 0$.

Proof. Given $c : C \to X$, $\left(\mathsf{h}^X(f) \circ \mathsf{h}^X(g)\right)(c) = c \circ g \circ f = 0$, since $g \circ f = 0$.

Final Step: $\mathrm{Ker}(\mathsf{h}^X(f)) = \mathrm{Im}(\mathsf{h}^X(g))$.

Proof. The last step already showed that $\mathrm{Im}(\mathsf{h}^X(g)) \subset \mathrm{Ker}(\mathsf{h}^X(f))$. Now we prove the converse. Let $b : B \to X$ be such that $\mathsf{h}^X(f)(b) = b \circ f = 0$. Since g is the cokernel of f, we have the following diagram:



which shows that $b = \overline{b} \circ g \in \mathrm{Im}(\mathsf{h}^X(g))$, as we wanted. □

(10.5) Example of Hom **functors.**– It is well–known that Hom functors are *not* right exact. Let us see the standard examples to demonstrate this fact. Consider the following short exact sequence in **Ab**:

$$0 \to \mathbb{Z} \xrightarrow{\mu_2} \mathbb{Z} \xrightarrow{p} \mathbb{Z}/2\mathbb{Z} \to 0,$$

where $\mu_2(a) = 2a$ is multiplication by 2.

Apply the covariant functor $\mathsf{h}_{\mathbb{Z}/2\mathbb{Z}} = \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, -)$ to get

$$0 \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to 0.$$

Note that there are no non–zero group homomorphisms from $\mathbb{Z}/2\mathbb{Z}$ to $\mathbb{Z}$, and $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Hence $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = \{0\}$, and the above sequence simplifies to:

$$0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0,$$

which is clearly not exact at $\mathbb{Z}/2\mathbb{Z}$.

In the contravariant case, let us apply $\mathsf{h}^{\mathbb{Z}} = \mathrm{Hom}(-, \mathbb{Z})$ to the same short exact sequence. We get:

$$0 \to \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{- \circ \mu_2} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \to 0.$$

Again, the sequence simplifies to

$$0 \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to 0,$$

and the last map is clearly not surjective.