LECTURE 11

(11.0) Left and right exact functors.– Recall that last time we introducted the notion of an exact sequence, and defined left/right exact functors $F : \mathcal{A} \to \mathcal{B}$ between two abelian categories $\mathcal{A}$ and $\mathcal{B}$. Thus, for every short exact sequence $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{A}$:

- *Covariant case.*

$$\textbf{Left exact:} \qquad 0 \to F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \text{ is exact.}$$

$$\textbf{Right exact:} \qquad F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \to 0 \text{ is exact.}$$

- *Contravariant case.*

$$\textbf{Left exact:} \qquad 0 \to G(Z) \xrightarrow{G(g)} G(Y) \xrightarrow{G(f)} G(X) \text{ is exact.}$$

$$\textbf{Right exact:} \qquad G(Z) \xrightarrow{G(g)} G(Y) \xrightarrow{G(f)} G(X) \to 0 \text{ is exact.}$$

We prove that for any $X \in \mathcal{A}$ (abelian category), h^X and h_X are both left exact functors $\mathcal{A} \to \mathbf{Ab}$. We saw some examples to demonstrate that they are not necessarily (right) exact.

(11.1) Tensor product I.– The most typical example of a functor which is right exact, but not necessarily left exact comes from *tensor products*. In this course, we will mostly be concerned with tensor product of modules over a commutative ring.

Let A be a unital, commutative ring. Let A-mod be the category of A–modules. It is an abelian category (see §9.4). Recall that for a commutative ring, we do not distinguish between left/right modules. Note that the Hom–functors for this category take values in A-mod again. That is, given $M, N \in A$-mod, $\mathrm{Hom}_A(M, N)$ is naturally an A–module, with scalar multiplication given by:

$$f \in \mathrm{Hom}_A(M, N), a \in A \qquad \rightsquigarrow \qquad (a \cdot f)(m) := f(am) = af(m), \ \forall\, m \in M.$$

Definition. Let M, N, P be three A–modules. An A*–bilinear map* $f : M \times N \to P$ is a set map such that (i) for every $m \in M$, $f(m, -) : N \to P$ is an A–module homomorphism, and (ii) for every $n \in N$, $f(-, n) : M \to P$ is an A–module homomorphism. More explicitly, the following equations must hold for every $a_1, a_2 \in A$, $m, m_1, m_2 \in M$ and $n, n_1, n_2 \in N$:

$$f(a_1m_1+a_2m_2, n) = a_1f(m_1, n)+a_2f(m_2, n), \qquad f(m, a_1n_1+a_2n_2) = a_1f(m, n_1)+a_2f(m, n_2).$$

Let us denote the set of all A–bilinear maps $M \times N \to P$ by $\mathrm{Bilinear}_A(M, N; P)$. This set also has a structure of an A–module (see Remark below).

Warning. $\mathrm{Bilinear}_A(M, N; P) \neq \mathrm{Hom}_A(M \times N, P)$. Think of the dot product as a typical example of an $\mathbb{R}$–bilinear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

Remark. We have the following equality:

$$\text{Bilinear}_A(M,N;P) = \text{Hom}_A(M, \text{Hom}_A(N,P)).$$

This makes it clear that the set $\text{Bilinear}_A(M,N;P)$ has an A–module structure. Also, note that for a bilinear $f : M \times N \to P$ and an A–linear $g : P \to Q$, the composition $g \circ f : M \times N \to Q$ is again A–bilinear.

$$\text{Bilinear}_A(M,N;P) \times \text{Hom}_A(P,Q) \xrightarrow{\circ} \text{Bilinear}_A(M,N;Q).$$

(11.2) Tensor product II.– Let $M, N \in A$-mod, and consider the following (covariant) functor:

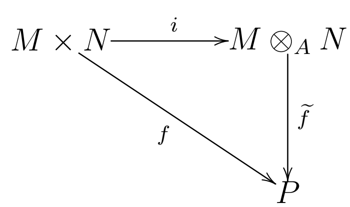
$$T_{M,N} : A\text{-mod} \to A\text{-mod}, \qquad P \mapsto T_{M,N}(P) := \text{Bilinear}_A(M,N;P).$$

On morphisms: $g : P \to Q$ is mapped to $T_{M,N}(g) = g \circ -$.

Definition. If $T_{M,N}$ is representable, the object representing it is denoted by $M \otimes_A N$ called *the tensor product of* M and N (over A). Often, we omit the subscript A, and just write $M \otimes N$, if the ring in question is clear from the context.

In more detail, $M \otimes_A N$ is an A–module together with an A–bilinear map $i : M \times N \to M \otimes_A N$, such that the following universal property holds.

Universal property of tensor product. For every $P \in A$-mod, and an A–bilinear map $f : M \times N \to P$, there exists a unique A–linear map $\widetilde{f} : M \otimes_A N \to P$ making the following diagram commute:



(11.3) Construction of tensor product.– Now we give a construction of tensor product, to make sure that it exists. Let $M, N \in A$-mod. Let $\widetilde{T}$ be the free A–module generated over the set of symbols $\{t(m,n) : m \in M, n \in N\}$. That is, a typical element of $\widetilde{T}$ is a finite linear expression:

$$\widetilde{T} = \left\{ \sum_{(m,n) \in M \times N} a_{m,n} t(m,n) : a_{m,n} \in A \text{ are all zero, except for finitely many} \right\}.$$

Thus, $\widetilde{T}$ is naturally an A–module. It is the direct sum of $M \times N$ many copies of A.

Let $K \subset \widetilde{T}$ be the A–submodule generated by following elements:

- $t(a_1m_1 + a_2m_2, n) - a_1t(m_1, n) - a_2t(m_2, n)$, for every $a_1, a_2 \in A$, $m_1, m_2 \in M$ and $n \in N$.

- $t(m, a_1n_1 + a_2n_2) - a_1t(m, n_1) - a_2t(m, n_2)$, for every $a_1, a_2 \in A$, $m \in M$ and $n_1, n_2 \in N$.

Define $T := \widetilde{T}/K$. We have an A–bilinear map $i : M \times N \to T$ sending $(m, n) \mapsto t(m, n) + K$.

Proposition. *T together with the A–bilinear map $i : M \times N \to T$ satisfies the universal property of the tensor product of M and N over A. It will be denoted by $M \otimes_A N$, after the proof is complete.*

Proof. Let $P \in A$-mod and let $f : M \times N \to P$ be an A–bilinear map. Viewing it as a set map, we get a unique A–linear map $F : \widetilde{T} \to P$ such that $F \circ i = f$. Here, we are using the same letter to denote the (set) map $M \times N \to \widetilde{T}$, defined as: $i(m, n) = t(m, n)$. The A–linear map F is thus given by: $F(t(m, n)) = f(m, n)$.

It remains to show that $F : \widetilde{T} \to P$ factors through K. That is $F(k) = 0$ for every $k \in K$. Note that since F is, by definition, A–linear, it suffices to check $F(k) = 0$, when k is one of the generators listed above.

When $k = t(a_1m_1 + a_2m_2, n) - a_1t(m_1, n) - a_2t(m_2, n)$, we get

$$\begin{aligned} F(k) &= F(t(a_1m_1 + a_2m_2, n)) - a_1F(t(m_1, n)) - a_2F(t(m_2, n)) \\ &= f(a_1m_1 + a_2m_2, n) - a_1f(m_1, n) - a_2f(m_2, n) = 0, \end{aligned}$$

by bilinearity of f. Similarly for the other type of generator of k, we can easily see that $F(k) = 0$. Thus, F factors through a unique morphism $\widetilde{f} : T \to P$, sending the coset $t(m, n) + K$ to $f(m, n)$. This proves that T satisfies the universal property written in §11.2 above. □

Remark. The module constructed in the previous section will be denoted by $M \otimes_A N$. We use the notation $m \otimes n$ to mean $i(m, n)$, where $i : M \times N \to M \otimes_A N$ is the canonical bilinear map. Informally speaking, $M \otimes_A N$ consists of finite linear expressions of the form $\sum_{i=1}^{p} m_i \otimes n_i$, which are manipulated according to the following rules:

- $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$, for every $m_1, m_2 \in M$ and $n \in N$.
- $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$, for every $m \in M$ and $n_1, n_2 \in N$.
- $(am) \otimes n = m \otimes (an)$, for every $m \in M$, $n \in N$ and $a \in A$.