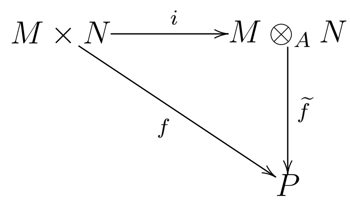
LECTURE 12

(12.0) Tensor products.– Recall that last time we defined and constructed the tensor product of two modules M and N over a commutative ring A, denoted by $M \otimes_A N$. Abstractly, it is the unique A–module which comes together with an A–bilinear map $i : M \times N \to M \otimes_A N$, satisfying a universal property:

For every $P \in A$-mod, and an A–bilinear map $f : M \times N \to P$, there exists a unique A–linear map $\widetilde{f} : M \otimes_A N \to P$ making the following diagram commute:



In less words, $M \otimes_A N$ is defined by:

$$\mathrm{Hom}_A(M \otimes_A N, P) \xrightarrow{\sim} \mathrm{Bilinear}_A(M, N; P).$$

Concretely, $M \otimes_A N$ consists of finite linear expressions of the form $\displaystyle\sum_{i=1}^{p} m_i \otimes n_i$, which are manipulated according to the following rules:

- $(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$, for every $m_1, m_2 \in M$ and $n \in N$.
- $m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$, for every $m \in M$ and $n_1, n_2 \in N$.
- $(am) \otimes n = m \otimes (an)$, for every $m \in M$, $n \in N$ and $a \in A$.

Now we will see some examples, and functorial properties of tensor products.

(12.1) Elementary properties of tensor product.– We will always use the universal property of tensor product to prove its properties.

Proposition.

(1) *For every $M, N \in A$-mod, $M \otimes_A N \cong N \otimes_A M$.*
(2) *$A \otimes_A N \cong N$ for every $N \in A$-mod.*
(3) *If $\mathfrak{a} \subset A$ is an ideal of A, then*

$$A/\mathfrak{a} \otimes_A N \cong N/\mathfrak{a}N.$$

Proof. (1). Consider the flip map $\sigma : M \times N \to N \times M$. Let $i : M \times N \to M \otimes_A N$ and $j : N \times M \to N \otimes_A M$ be the canonical A–bilinear maps. It is clear that $j \circ \sigma : M \times N \to N \otimes_A M$ is A–bilinear, hence gives rise to an A–linear map $f : M \otimes_A N \to N \otimes_A M$ given by: $f(m \otimes n) = n \otimes m$. Similarly, using $i \circ \sigma^{-1} : N \times M \to M \otimes_A N$, we conclude that there is a unique A–linear map $g : N \otimes_A M \to M \otimes_A N$, given by $g(n \otimes m) = m \otimes n$. Clearly f and g are inverse to each other, hence $M \otimes_A N \cong N \otimes_A M$.

(2). Note that if $f : A \times N \to P$ is any A–bilinear map, then we get an A–linear map $\overline{f} : N \to P$ as $\overline{f}(n) = f(1, n)$. The bilinear map f is uniquely determined by $\overline{f}$, since $f(a, n) = af(1, n) = a\overline{f}(n) = \overline{f}(an)$. Hence,

$$\text{Bilinear}_A(A, N; P) \xrightarrow{\sim} \text{Hom}_A(N, P), \qquad f \mapsto \overline{f}.$$

So, N satisfies the universal property of $A \otimes_A N$.

(3). The argument is same as the one for (2). Given a bilinear map $f : A/\mathfrak{a} \times N \to P$, we use it to define $\overline{f} : N \to P$ as $\overline{f}(n) = f(1 + \mathfrak{a}, n)$. It remains to be shown that $\overline{f}$ factors through the submodule

$$\mathfrak{a}N = \{an : a \in \mathfrak{a},\ n \in N\} \subset N.$$

That is, $\overline{f}(an) = 0$ for every $a \in \mathfrak{a}$ and $n \in N$. This is true because (we are using $\overline{x} = x + \mathfrak{a} \in A/\mathfrak{a}$):

$$\overline{f}(an) = f(\overline{1}, an) = af(\overline{1}, n) = f(\overline{a}, n) = f(0, n) = 0.$$

Denoting by the same symbol, the resulting A–linear map $\overline{f} : N/\mathfrak{a}N \to P$, we obtain the isomorphism (same argument as before, hence omitted):

$$\text{Bilinear}_A(A/\mathfrak{a}, N; P) \xrightarrow{\sim} \text{Hom}_A(N/\mathfrak{a}N, P), \qquad f \mapsto \overline{f},$$

which proves (3). □

(12.2) Examples.– The following examples of tensor product follow immediately from Proposition 12.1.

I. Let $m, n \in \mathbb{Z}_{\geq 2}$. Then $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z}$. This is because we use (3) of the proposition above, and:

$$(\mathbb{Z}/n\mathbb{Z})/m(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/(m, n) = \mathbb{Z}/\gcd(m, n)\mathbb{Z}.$$

Here (m, n) is the ideal of $\mathbb{Z}$ generated by m and n, which is same as $\gcd(m, n)\mathbb{Z}$ by Euclidean algorithm.

II. Let K be a field and consider the commutative ring $K[x, y]$. Let $K[x]$ be viewed as an A–module via $K[x] \cong K[x, y]/(y)$ (similarly $K[y]$). Then:

$$K[x] \otimes_{K[x,y]} K[y] \cong K = K[x, y]/(x, y).$$

III. If $A = K$ is a field, $M, N \in A$-mod are K–vector spaces. Let $\{m_i\}_{i \in I}$ and $\{n_j\}_{j \in J}$ be bases of M and N resp. Then $M \otimes_K N$ is the vector space with basis $\{m_i \otimes n_j\}_{(i,j) \in I \times J}$.

(12.3) Functorial properties of tensor product.– Again, let A be a unital commutative ring.

Lemma. *Given A–linear morphisms $M \xrightarrow{f} M'$ and $N \xrightarrow{g} N'$, there is a unique A–linear map $f \otimes g : M \otimes N \to M' \otimes N'$ such that $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$.*

Moreover, if $M' \xrightarrow{f'} M''$ and $N' \xrightarrow{g'} N''$ are two more morphisms, then:

$$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g).$$

The proof is left as an exercise. All you have to do is apply the universal property to the composition:

$$M \times N \xrightarrow{(f,g)} M' \times N' \xrightarrow{i'} M' \otimes N'$$

where i' is the canonical bilinear map $M' \times N' \to M' \otimes N'$.

Using this lemma, we are able to view tensoring with a fixed module as a functor. Let $N \in A$-mod be fixed.

Corollary. *We have an additive functor* $- \otimes N : A\text{-mod} \to A\text{-mod}$.

Proof. This functor maps $M \in A$-mod to $M \otimes N$, and a morphism $f : M \to M'$ is mapped to $f \otimes \mathrm{Id}_N$. By the lemma above, it is clear that $\mathrm{Id}_M \otimes \mathrm{Id}_N = \mathrm{Id}_{M\otimes N}$, and that $(g \circ f) \otimes \mathrm{Id}_N = (g \otimes \mathrm{Id}_N) \circ (f \otimes \mathrm{Id}_N)$, proving that $- \otimes N$ is a functor.

To prove that it is additive, we need to show that

$$- \otimes \mathrm{Id}_N : \mathrm{Hom}_A(M_1, M_2) \to \mathrm{Hom}_A(M_1 \otimes N, M_2 \otimes N)$$

is a group homomorphism. It is clear by the lemma that $0 \otimes \mathrm{Id}_N = 0$. Now let $f, g : M_1 \to M_2$. Then for every $m \in M$ and $n \in N$, we have:

$$((f_1 + f_2) \otimes \mathrm{Id}_N)\ (m \otimes n) = (f_1(m) + f_2(m)) \otimes n,$$

$$((f_1 \otimes \mathrm{Id}_N) + (f_2 \otimes \mathrm{Id}_N))\ (m \otimes n) = f_1(m) \otimes n + f_2(m) \otimes n.$$

which are equal. Hence $(f_1 + f_2) \otimes \mathrm{Id}_N = (f_1 \otimes \mathrm{Id}_N) + (f_2 \otimes \mathrm{Id}_N)$. □

(12.4) Tensor-Hom adjointness.– The way we defined the tensor product, the following isomorphism is clear:

$$\mathrm{Hom}_A(M \otimes N, P) \xrightarrow{\sim} \mathrm{Bilinear}_A(M, N; P) = \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)).$$

Let us keep $N \in A$-mod fixed, and denote the inverse of this bijection by $\beta_{M,P}$ (to be consistent with our conventions). Note that for any $\xi : M \to \mathrm{Hom}_A(N, P)$, $\beta_{M,P}(\xi)$ is given by:

$$\beta_{M,P}(\xi) : m \otimes n \mapsto \xi(m)(n).$$

Proposition. $\beta_{M,P}$ *is natural in* M *and* P*. Hence, we have a pair of adjoint functors* $(- \otimes N, \mathrm{Hom}_A(N, -))$.

Proof. We only need to check that given any two morphisms $M \xrightarrow{f} M'$ and $P' \xrightarrow{g} P$ the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}_A(M, \mathrm{Hom}_A(N, P)) & \xrightarrow{\beta_{M,P}} & \mathrm{Hom}_A(M \otimes N, P) \\
\uparrow L & & \uparrow R \\
\mathrm{Hom}_A(M', \mathrm{Hom}_A(N, P')) & \xrightarrow[\beta_{M',P'}]{} & \mathrm{Hom}_A(M' \otimes N, P')
\end{array}$$

So, let $\xi : M' \to \mathrm{Hom}(N, P')$ be given. Following it through the composition $\beta_{M,P} \circ L$ and evaluating on $m \otimes n$ gives:

$$\beta_{M,P}(L(\xi))(m \otimes n) = L(\xi)(m)(n) = g(\xi(f(m))(n)).$$

Similarly, $R \circ \beta_{M',P'}$ sends ξ to the morphism which evaluated on $m \otimes n$ yields:

$$R(\beta_{M',P'}(\xi))(m \otimes n) = g(\beta_{M',P'}(\xi)(f(m) \otimes n)) = g(\xi(f(m))(n)).$$

This finishes the proof of naturality of β. □

(12.5) Exactness of tensor functor.– The following simple example shows that tensor is not left exact.

Example. Consider the short exat sequence of abelian groups ($\mathbb{Z}$–modules):

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0.$$

Tensoring with $- \otimes \mathbb{Z}/2\mathbb{Z}$ yields (recall $A \otimes_A N \cong N$):

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{0} \mathbb{Z}/2\mathbb{Z} \xrightarrow{\mathrm{Id}} \mathbb{Z}/2\mathbb{Z} \to 0$$

which is not exact at the first $\mathbb{Z}/2\mathbb{Z}$. Basically, tensoring could turn an injective morphism to zero morphism, which is the reason it is not left exact.

Theorem. *For any* $N \in A\text{-mod}$, $- \otimes_A N : A\text{-mod} \to A\text{-mod}$ *is right exact.*

Proof. We need to prove that given an exact sequence $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ of A–modules, the following sequence is also exact:

$$M_1 \otimes N \xrightarrow{f \otimes \mathrm{Id}} M_2 \otimes N \xrightarrow{g \otimes \mathrm{Id}} M_3 \otimes N \longrightarrow 0$$

$g \otimes \mathrm{Id}$ *is surjective.* This is clear since each simple tensor $m_3 \otimes n \in M_3 \otimes N$ is of the form $g(m_2) \otimes n = (g \otimes \mathrm{Id})(m_2 \otimes n)$ for some $m_2 \in M_2$, since g is assumed to be surjective. Hence, $m_3 \otimes n \in \mathrm{Im}(g \otimes \mathrm{Id})$. As $M_3 \otimes N$ is generated as an A–module by such element we conclude that $g \otimes \mathrm{Id}$ is surjective.

$\mathrm{Im}(f \otimes \mathrm{Id}) \subset \mathrm{Ker}(g \otimes \mathrm{Id})$. This is again clear, since by Lemma 12.3, $(g \otimes \mathrm{Id}) \circ (f \otimes \mathrm{Id}) = (g \circ f) \otimes \mathrm{Id} = 0$.

It remains to show that $\mathrm{Ker}(g \otimes \mathrm{Id}) = \mathrm{Im}(f \otimes \mathrm{Id})$. For this, note that $g \otimes \mathrm{Id}$ gives rise to a well–defined A–linear map $M_2 \otimes N / \mathrm{Im}(f \otimes \mathrm{Id}) \to M_3 \otimes N$, by the previous part, which we will denote by $\widetilde{g}$. We are going to construct an A–linear map $\widetilde{h} : M_3 \otimes N \to M_2 \otimes N / \mathrm{Im}(f \otimes \mathrm{Id})$ and show that $\widetilde{g}$ and h are inverse to each other.

Consider $h : M_3 \times N \to M_2 \otimes N / \mathrm{Im}(f \otimes \mathrm{Id})$ given by

$$h(m_3, n) = m_2 \otimes n \ (\mathrm{mod}\ \mathrm{Im}(f \otimes \mathrm{Id})),$$

where $m_2 \in M_2$ is chosen so that $g(m_2) = m_3$. Note that a different choice m_2' will differ from m_2 by an element of $\mathrm{Ker}(g) = \mathrm{Im}(f)$, so modulo $\mathrm{Im}(f \otimes \mathrm{Id})$, $m_2 \otimes n \equiv m_2' \otimes n$. This proves that h is well–defined.

It is left to the reader to check that h is A–bilinear, and hence gives rise to $\widetilde{h} : M_3 \otimes N \to M_2 \otimes N/\operatorname{Im}(f \otimes \mathrm{Id})$. By their definition, one can easily verify that $\widetilde{g}$ and $\widetilde{h}$ are inverse to each other. Hence, $\widetilde{g}$ sets up an isomorphism

$$\widetilde{g} : M_2 \otimes N/\operatorname{Im}(f \otimes \mathrm{Id}) \xrightarrow{\sim} M_3 \otimes N,$$

proving that $\operatorname{Im}(f \otimes \mathrm{Id})$ is indeed the kernel of $g \otimes \mathrm{Id}$. ☐

(12.6) A more hands-off proof.– Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories, and assume that we have a pair of adjoint functors (F, G): $\mathcal{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$.

Exercise. Prove that F is right exact, and G is left exact.

Thus, Theorem 12.5 can be obtained from “Tensor-Hom adjointness” Proposition 12.4.