LECTURE 13

(13.0) Overview of these notes.– In this set of notes we will discuss *derived functors*. A brilliant observation made in Grothendieck's Tôhoku paper was to view numerous cohomology (or homology) theories as sequence of functors *derived* from a left (or right) exact functor.

There are **a lot** of cohomology theories in mathematics. For example, de Rham cohomology (for manifolds), sheaf cohomology and its special cases: ℓ–adic cohomology, étale cohomology (for algebraic schemes), intersection cohomology (for singular spaces), Lie algebra cohomology (Chevalley–Eilenberg cohomology), group cohomology, Galois cohomology, cyclic cohomology... the list goes on.

One systematic way to view them is via the machinery of derived functors:

$$\text{Left exact functor } F \rightsquigarrow \boxed{\text{Derived functors}} \rightsquigarrow \begin{array}{c}\text{Sequence of functors}\\ \{R^nF\}_{n\geq 0}\end{array}$$

satisfying some naturally imposed axioms (analogous to long exact sequence in cohomology). An introduction to this axiomatic is the purpose of these notes. The material of these notes is for inspirational purposes only, and shows how an attempt to construct derived functors leads to the structures we will study in the next few weeks (see the "To do list" in the last paragraph).

(13.1) Right derived functors of a left exact covariant functor.– Assume $F : \mathcal{A} \to \mathcal{B}$ is an additive, covariant, left exact functor. *Right derived functors* of F is a sequence of (additive) covariants functors $\{R^nF : \mathcal{A} \to \mathcal{B}\}_{n\geq 0}$ together with the following data:

- A natural isomorphism $F \xrightarrow{\sim} R^0F$. We often just identify the two and write F for R^0F.
- For every short exact sequence $S : 0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, morphisms $\left\{\delta^n(S) : R^nF(Z) \to R^{n+1}F(X)\right\}_{n\geq 0}$ in $\mathcal{B}$ (called *connecting morphisms*).

This data is subject to two axioms:

(1) For every short exact sequence $S : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{A}$, the following (long) sequence is exact:

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \xrightarrow{\delta^0(S)} R^1F(X) \xrightarrow{R^1F(f)} R^1F(Y) \xrightarrow{R^1F(g)} R^1F(Z) \xrightarrow{\delta^1(S)} R^2F(X) \xrightarrow{R^2F(f)} R^2F(Y) \xrightarrow{R^2F(g)} R^2F(Z) \cdots$$

(2) If we have the following commutative diagram, with exact rows in $\mathcal{A}$, denoted by S_1 and S_2 respectively,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{g_1} & Z_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} & & \\ 0 & \longrightarrow & X_2 & \xrightarrow{f_2} & Y_2 & \xrightarrow{g_2} & Z_2 & \longrightarrow & 0 \end{array}$$

then, the following diagram commutes, for every $n \geq 0$:

$$\begin{array}{ccc} R^nF(Z_1) & \xrightarrow{\delta^n(S_1)} & R^{n+1}F(X_1) \\ \downarrow{\scriptstyle R^nF(z)} & & \downarrow{\scriptstyle R^{n+1}F(x)} \\ R^nF(Z_2) & \xrightarrow[\delta^n(S_2)]{} & R^{n+1}F(X_2) \end{array}$$

(13.2) Contravariant, left exact functor.– If $F : \mathcal{A} \to \mathcal{B}$ is *contravariant* additive, left exact functor, its right derived functors is a sequence of contravariant functors $\{R^nF : \mathcal{A} \to \mathcal{B}\}_{n\geq 0}$, together with

- An identification of R^0F with F.
- For every short exact sequence $S : 0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, connecting morphisms $\left\{\delta^n(S) : R^nF(X) \to R^{n+1}F(Z)\right\}_{n\geq 0}$ in $\mathcal{B}$.

Again we have two axioms analogous to the ones from the previous paragraphs.

(1) For every short exact sequence $S : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{A}$, the following (long) sequence is exact:

$$0 \longrightarrow F(Z) \xrightarrow{F(g)} F(Y) \xrightarrow{F(f)} F(X) \xrightarrow{\delta^0(S)} R^1F(Z) \xrightarrow{R^1F(g)} R^1F(Y) \xrightarrow{R^1F(f)} R^1F(X) \xrightarrow{\delta^1(S)} R^2F(Z) \xrightarrow{R^2F(g)} R^2F(Y) \xrightarrow{R^2F(f)} R^2F(X) \cdots$$

(2) If we have the following commutative diagram, with exact rows in $\mathcal{A}$, denoted by S_1 and S_2 respectively,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{g_1} & Z_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} & & \\ 0 & \longrightarrow & X_2 & \xrightarrow{f_2} & Y_2 & \xrightarrow{g_2} & Z_2 & \longrightarrow & 0 \end{array}$$

then, the following diagram commutes, for every $n \geq 0$:

$$\begin{array}{ccc} R^nF(X_1) & \xrightarrow{\delta^n(S_1)} & R^{n+1}F(Z_1) \\ \uparrow{\scriptstyle R^nF(x)} & & \uparrow{\scriptstyle R^{n+1}F(z)} \\ R^nF(X_2) & \xrightarrow[\delta^n(S_2)]{} & R^{n+1}F(Z_2) \end{array}$$

(13.3) Covariant, right exact case.– Let $G : \mathcal{A} \to \mathcal{B}$ be a covariant, additive, right exact functor. Its *left derived functors* consist of a sequence of (additive) covariant functors $\{L_nG : \mathcal{A} \to \mathcal{B}\}_{n\geq 0}$, together with the data of:

- An identification of G with L_0G.
- For every short exact sequence $S : 0 \to X \to Y \to Z \to 0$ in $\mathcal{A}$, connecting morphisms $\{\delta_n(S) : L_{n+1}G(Z) \to L_nG(X)\}_{n\geq 0}$.

Again there are two axioms. In the case of right exact functors, the long exact sequence continues indefinitely to the right side.

(1) For every short exact sequence $S : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ in $\mathcal{A}$, the following (long) sequence is exact:

$$\cdots \longrightarrow L_2G(X) \xrightarrow{L_2G(f)} L_2G(Y) \xrightarrow{L_2G(g)} L_2G(Z) \xrightarrow{\delta_1(S)} L_1G(X) \xrightarrow{L_1G(f)} L_1G(Y) \xrightarrow{L_1G(g)} L_1G(Z) \xrightarrow{\delta_0(S)} G(X) \xrightarrow{G(f)} G(Y) \xrightarrow{G(g)} G(Z) \longrightarrow 0$$

(2) If we have the following commutative diagram, with exact rows in $\mathcal{A}$, denoted by S_1 and S_2 respectively,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \xrightarrow{f_1} & Y_1 & \xrightarrow{g_1} & Z_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} & & \downarrow{\scriptstyle z} & & \\ 0 & \longrightarrow & X_2 & \xrightarrow{f_2} & Y_2 & \xrightarrow{g_2} & Z_2 & \longrightarrow & 0 \end{array}$$

then, the following diagram commutes, for every $n \geq 0$:

$$\begin{array}{ccc} L_{n+1}G(Z_1) & \xrightarrow{\delta_n(S_1)} & L_nG(X_1) \\ \downarrow{\scriptstyle L_{n+1}G(z)} & & \downarrow{\scriptstyle L_nG(x)} \\ L_{n+1}G(Z_2) & \xrightarrow[\delta_n(S_2)]{} & L_nF(X_2) \end{array}$$

(13.4) Some heuristic discussions.– To fix ideas, let us assume $F : \mathcal{A} \to \mathcal{B}$ is covariant and left exact. Let us try to see if our definition constraints what $R^1F : \mathcal{A} \to \mathcal{B}$ could possibly be. That is, if we can define $R^1F(X)$ for $X \in \mathcal{A}$.

By requirement of long exact sequence, we know that if $0 \to X \to Y \to Z \to 0$ is exact in $\mathcal{A}$, then the following sequence is supposed to be exact:

$$0 \to F(X) \to F(Y) \to F(Z) \to R^1F(X) \to R^1F(Y) \to \cdots$$

So, if we picked Y "cleverly" so that $R^1F(Y)$ has to be 0 (we will be more precise later), then $R^1F(X) = \mathrm{CoKer}(F(Y) \to F(Z))$ by the required exactness.

What does it mean for an object Y, to vanish under R^1F? It means that no matter what short exact sequence starting from Y we pick: $0 \to Y \to \widetilde{Y} \to U \to 0$, the following sequence is exact:

$$0 \to F(Y) \to F(\widetilde{Y}) \to F(U) \to 0 = R^1F(Y).$$

Now we straighten our logic.

(1) *Definition.* An object $Y \in \mathcal{A}$ is called *F–acyclic* (or, F–exact) if for every short exact sequence $0 \to Y \to \widetilde{Y} \to U \to 0$, the following sequence is exact:

$$0 \to F(Y) \to F(\widetilde{Y}) \to F(U) \to 0.$$

(2) *Assumption.* $\mathcal{A}$ has enough F–acyclic objects. Meaning, for every $X \in \mathcal{A}$, there exists some F–acyclic Y and an injective morphism $f : X \to Y$.

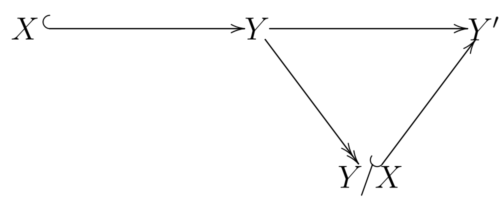
(3) *Construction.* For $X \in \mathcal{A}$, choose an injective morphism $f : X \to Y$, where Y is F–acyclic. Let $\pi : Y \to Y/X$ denote the cokernel of f, and define:

$$\boxed{R^1F(X) := \text{CoKer}(F(\pi) : F(Y) \to F(Y/X)).}$$

(Hope for the best: that this definition does not depend on the choice of Y).

(13.5) Higher derived functors.– If all the wishes from the previous paragraph come true, we will be able to define higher derived functors inductively as follows. Given $X \in \mathcal{A}$, find an injective morphism $X \to Y$, where Y is F–acyclic. Set $R^{k+1}F(X) = R^kF(Y/X)$, for every $k \geq 1$.

For instance, in order to get $R^2F(X)$, we will have to compute $R^1F(Y/X)$. Denoting Y/X by $\overline{Y}$, this was done by choosing another F–acyclic Y', an injective map $\overline{Y} \to Y'$, and taking the cokernel of $F(Y') \to F(Y'/\overline{Y})$.



This line of reasoning leads us to the notion of an *F–acyclic resolution.*

Definition. Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact, covariant functor. Given $X \in \mathcal{A}$, a *resolution* of X by F–acyclic objects is a sequence of morphisms:

$$Y^\bullet : \qquad Y^0 \xrightarrow{f^0} Y^1 \xrightarrow{f^1} \cdots$$

such that:

- $\text{Ker}(f^0) = X$.
- The sequence is exact at Y^n, for every $n \geq 1$.

The logic worked out preceeding this definition yields the following recipe for defining $\{R^nF(X)\}_{n\geq 0}$. Choose an F–acyclic resolution $Y^\bullet$ of X as above. Apply F to it (as F is additive, we are only promised to get a complex, not necessary exact sequence):

$$0 \to F(Y^0) \xrightarrow{F(f^0)} F(Y^1) \xrightarrow{F(f^1)} F(Y^2) \xrightarrow{F(f^2)} \cdots$$

and define (with the convention $F(f^{-1}) = 0$):

$$\boxed{R^nF(X) := \mathrm{Ker}(F(f^n))/\,\mathrm{Im}(F(f^{n-1})) = H^n(F(Y^\bullet))}$$

Here, we are borrowing notation from algebraic topology: for any complex $Z^\bullet : 0 \to Z^0 \to Z^1 \to \cdots$ in an abelian category, its n^{th} cohomology is defined as:

$$\boxed{H^n(Z^\bullet) := \frac{\mathrm{Ker}(Z^n \to Z^{n+1})}{\mathrm{Im}(Z^{n-1} \to Z^n)}}$$

Remark. At this point, nothing is obviously true. We don't know that the thing defined above on objects is a functor. We don't know how the connecting morphisms will be defined.

(13.6) To do list.– Let us pause and reflect on the amount of work laid out before us. This "to do list" contains precisely the topics we will cover in this part of the course (i.e, Homological Algebra).

(1) Study category of complexes and functorial properties of H^n : Complexes over $\mathcal{A} \to \mathcal{A}$. In this part, we will encounter:
   - The famous "snake lemma" which gives rise to the connecting morphisms.
   - The notion of *homotopy* between two morphisms of complexes, and how homotopic morphisms induce the same morphism under H^n. This will be needed to prove that our construction is *independent of the chosen resolution.*

(2) Study F–acyclic objects. As we will see, when we get to this part, that:
   - *Injective objects* are F–acyclic for any (i) left exact, covariant or (ii) right exact contravariant F.
   - *Projective objects* are F–acylic for any (i) left exact contravariant, or (ii) right exact covariant F.

(3) Prove uniqueness (up to homotopy) of injective and projective resolutions.

Once we reach the end of this long road, we will be able to define Ext (and Tor) as right (and left) derived functors of Hom (and $\otimes$). At that point, we will let go of the abstraction and see how to compute these things explicitly.