LECTURE 14

(14.0) Category of complexes.– Let R be a ring and let $\mathcal{A} = R$-mod be the abelian category of left R–modules. All the constructions and results of these notes work for *any abelian category*. However, to keep things concrete (i.e, so that we talk about "elements" of an object), we are going to narrow our attention to $\mathcal{A} = R$-mod.

Definition. The category of cochain complexes over $\mathcal{A}$, denoted by $\mathbb{K}^\bullet(\mathcal{A})$, consists of following objects.

$$(C^\bullet, d^\bullet) : \qquad \cdots \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \xrightarrow{d^{n+1}} \cdots$$

where,

- $C^n \in \mathcal{A}$ for every $n \in \mathbb{Z}$.
- $d^{n+1} \circ d^n = 0$ for every $n \in \mathbb{Z}$.

Morphisms from an object $(C^\bullet, d_C^\bullet)$ to $(D^\bullet, d_D^\bullet)$ consist of $\{f^n : C^n \to D^n\}_{n\in\mathbb{Z}}$ such that $d_D^n \circ f^n = f^{n+1} \circ d_C^n$.

Remarks.

(1) The terminology is again borrowed from algebraic topology. For chain complexes, we put numbers in the subscript and they decrease from left to right. Their category is often denoted by $\mathbb{K}_\bullet(\mathcal{A})$. There is no conceptual difference between the two setting - merely notational.

(2) In some texts, it is assumed that cochain complexes are indexed by $\mathbb{Z}_{\geq 0}$ (i.e, they are bounded from left). When we get to injective resolutions, this will be the case, but for now our cochains are unbounded on both sides.

(3) *Abuse of notation.* The morphisms d^n in the definition are called *differentials*. It is customary to drop the superscripts and write $d \circ d = 0$, if the index of the domain is implicitly clear. Similarly, we just say "let $X^\bullet$ be a complex", instead of $(X^\bullet, d_X^\bullet)$ to save some space.

(14.1) Category $\mathbb{K}^\bullet(\mathcal{A})$ continued.–

Proposition. *The category $\mathbb{K}^\bullet(\mathcal{A})$ is abelian.*

This proposition is proved using an argument similar to the one in §9.5. We merely record some of the observations made there, in the context of $\mathbb{K}^\bullet(\mathcal{A})$.

For two complexes $X^\bullet$ and $Y^\bullet$ in $\mathbb{K}^\bullet(\mathcal{A})$, the set of homomorphisms

$$\mathrm{Hom}_{\mathbb{K}^\bullet(\mathcal{A})}(X^\bullet, Y^\bullet) \subset \prod_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{A}}(X^n, Y^n),$$

is defined by the condition $d_Y^n \circ \alpha^n = \alpha^{n+1} \circ d_X^n$. So, it is naturally a subgroup.

For a morphism $\alpha^\bullet : X^\bullet \to Y^\bullet$, we define its kernel, image etc. component–wise. For instance, $K^\bullet = \mathrm{Ker}(\alpha^\bullet)$ is the following complex.

- $K^n = \mathrm{Ker}(\alpha^n : X^n \to Y^n)$.
- By the commutativity of the following diagram, the differential of $X^\bullet$ restricts to $K^\bullet$:

$$\begin{array}{ccccc} K^n & \longrightarrow & X^n & \xrightarrow{\alpha^n} & Y^n \\ \downarrow & & \downarrow{\scriptstyle d_X^n} & & \downarrow{\scriptstyle d_Y^n} \\ K^{n+1} & \longrightarrow & X^{n+1} & \xrightarrow{\alpha^{n+1}} & Y^{n+1} \end{array}$$

This type of argument was implicitly used in §9.6, but let us record it as a lemma for future use:

Lemma. *Let $\mathcal{C}$ be an abelian category, and assume that we have a commutative diagram in $\mathcal{C}$:*

$$\begin{array}{ccc} A_1 & \xrightarrow{a} & A_2 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B_1 & \xrightarrow[b]{} & B_2 \end{array}$$

Then, we get a morphism $f' : \mathrm{Ker}(a) \to \mathrm{Ker}(b)$ and a morphism $g' : \mathrm{CoKer}(a) \to \mathrm{CoKer}(b)$ making the following diagram commute.

$$\begin{array}{ccccccc} \mathrm{Ker}(a) & \longrightarrow & A_1 & \xrightarrow{a} & A_2 & \longrightarrow & \mathrm{CoKer}(a) \\ {\scriptstyle f'}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ \mathrm{Ker}(b) & \longrightarrow & B_1 & \xrightarrow[b]{} & B_2 & \longrightarrow & \mathrm{CoKer}(b) \end{array}$$

Proof. Let us see how f' arises. Consider the composition

$$\mathrm{Ker}(a) \to A_1 \xrightarrow{f} B_1 \xrightarrow{b} B_2.$$

This composition is zero, since it is same as the following, since $b \circ f = g \circ a$, which is zero by definition of the kernel.

$$\mathrm{Ker}(a) \to A_1 \xrightarrow{a} A_2 \xrightarrow{g} B_2.$$

Hence, by the definition of $\mathrm{Ker}(b)$, the composition $\mathrm{Ker}(a) \to A_1 \to B_1$ factors through $\mathrm{Ker}(b)$ giving rise to $f' : \mathrm{Ker}(a) \to \mathrm{Ker}(b)$ making the left square in the diagram above commute. Same argument works for the cokernels. □

Direct sum of two complexes $X^\bullet$ and $Y^\bullet$ is also defined term-wise: $(X^\bullet \oplus Y^\bullet)^n := X^n \oplus Y^n$, with differential (d_X^n, d_Y^n).

(14.2) Cocycles, coboundaries and cohomology.– Given a complex $X^\bullet$ in $\mathbb{K}^\bullet(\mathcal{A})$, and an integer $n \in \mathbb{Z}$, we define:

$$\mathcal{Z}^n(X^\bullet) := \mathrm{Ker}(d^n : X^n \to X^{n+1}) \qquad n\text{–cocyles.}$$

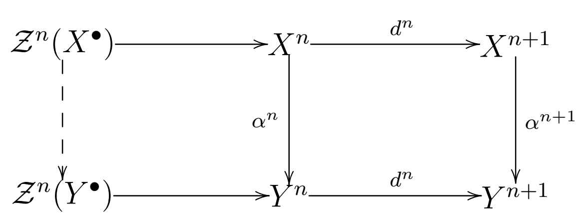
$$\mathcal{B}^n(X^\bullet) := \mathrm{Im}(d^{n-1} : X^{n-1} \to X^n) \qquad n\text{–coboundaries.}$$

Note that since $d \circ d = 0$, we have: $\mathcal{B}^n(X^\bullet) \subset \mathcal{Z}^n(X^\bullet)$. The n^{th} *cohomology* of $X^\bullet$ is then defined as the quotient:

$$\boxed{H^n(X^\bullet) = \mathcal{Z}^n(X^\bullet)/\mathcal{B}^n(X^\bullet) = \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1})}$$

Proposition. *For every $n \in \mathbb{Z}$, $\mathcal{Z}^n, \mathcal{B}^n, H^n$ are additive, covariant functors $\mathbb{K}^\bullet(\mathcal{A}) \to \mathcal{A}$.*

Proof. Let us first see it for the functor of cocycles. We know how to define it on objects. For a morphism $\alpha^\bullet : X^\bullet \to Y^\bullet$, we have the following commutative diagram (here $n \in \mathbb{Z}$ is fixed).



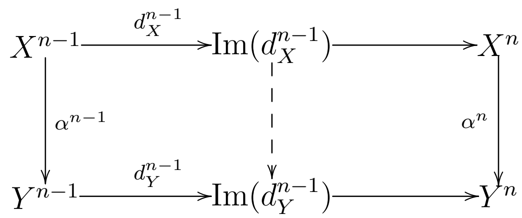
This follows from the lemma in §14.1 above. The argument using elements of X^n goes as follows, to show that α^n restricts to a morphism $\mathcal{Z}^n(X^\bullet) \to \mathcal{Z}^n(Y^\bullet)$. Let $x \in \mathcal{Z}^n(X^\bullet)$, that is, $d^n(x) = 0$. Thus, $d^n(\alpha^n(x)) = \alpha^{n+1}(d^n(x)) = 0$ proving that $\alpha^n(x) \in \mathcal{Z}^n(Y^\bullet)$.

This allows us to define $\mathcal{Z}^n(\alpha^\bullet)$ as the restriction of α^n to n–cocyles, which we continue to denote by α^n. This description makes it clear that:

$$\mathcal{Z}^n(\mathrm{Id}_{X^\bullet}) = \mathrm{Id}_{\mathcal{Z}^n(X^\bullet)}, \qquad \mathcal{Z}^n(\beta^\bullet \circ \alpha^\bullet) = \mathcal{Z}^n(\beta^\bullet) \circ \alpha^\bullet.$$

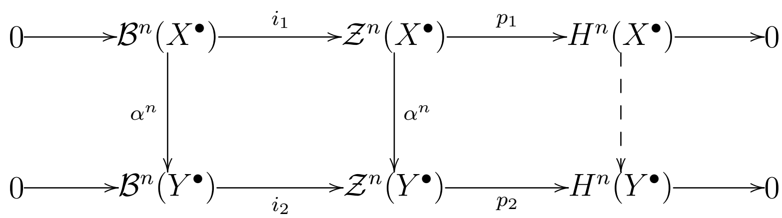
That is, $\mathcal{Z}^n$ is a functor. It is also easily verified that it is additive on morphisms, since it is merely a restriction of the domain of the morphisms, which respects addition.

The same argument as before works for $\mathcal{B}^n$. Namely, we have the following diagram that defines $\mathcal{B}^n(\alpha^\bullet)$ as the restriction of α^n to $\mathrm{Im}(d_X^{n-1}) \subset X^n$.



(apply Lemma 14.1 twice, first to get a map from $\mathrm{CoKer}\, d_X^{n-1} \to \mathrm{CoKer}\, d_Y^{n-1}$, and then with $A_1 = X^n \to A_2 = \mathrm{CoKer}\, d_X^{n-1}$, $B_1 = Y^n \to B_2 = \mathrm{CoKer}(d_Y^{n-1})$.)

Finally $H^n(\alpha^\bullet)$ is defined via the following diagram, where the rows are short exact sequences (by definition).



This assertion again follows from the lemma in §14.1 above. A direct argument with "elements" is given here, to see clearly how the definition of the dashed arrow works.

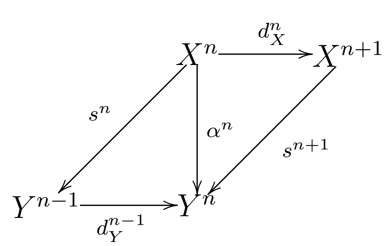
For $x \in H^n(X^\bullet)$, choose $\widetilde{x} \in \mathcal{Z}^n(X^\bullet)$ such that $p_1(\widetilde{x}) = x$. Let $\widetilde{y} = \alpha^n(\widetilde{x})$ and map $x \mapsto p_2(\widetilde{y}) \in H^n(Y^\bullet)$. To see that it is well–defined, if we chose another $\widetilde{x}'$, then $p_1(\widetilde{x}-\widetilde{x}') = 0$ implying that $\widetilde{x} = \widetilde{x}' + i_1(b)$ for some $b \in \mathcal{B}^n(X^\bullet)$. Thus, $\widetilde{y} = \widetilde{y}' + i_2(\alpha^n(b))$. As $p_2 \circ i_2 = 0$, we get that $p_2(\widetilde{y}) = p_2(\widetilde{y}')$. Note that we didn't need i_1 and i_2 to be injective here.

Remark. With the "concrete proof", it remains to show that the resulting map is R–linear, which is left to the reader. With the "hands–off proof", we lose a bit of clarity, but the gain is that we don't have to check anything. $\square$

Exercise. Show that $\mathcal{Z}^n$ is left exact.

(14.3) Null–homotopic morphisms.– Let $\alpha^\bullet : X^\bullet \to Y^\bullet$ be a morphism of two complexes. We say $\alpha^\bullet$ is *null–homotopic* if there exists morphisms $s^n : X^n \to Y^{n-1}$ such that

$\boxed{\alpha = ds + sd}$ (with appropriate superscripts - see the diagram below)



As usual, two morphisms $\alpha^\bullet, \beta^\bullet : X^\bullet \to Y^\bullet$ are said to be *homotopic* if $\alpha^\bullet - \beta^\bullet$ is null–homotopic. In symbols, we write $\alpha^\bullet \sim \beta^\bullet$ to mean $\alpha^\bullet$ and $\beta^\bullet$ are homotopic.

Proposition. *The set of null–homotopic morphisms* $\mathrm{Hom}^0(X^\bullet, Y^\bullet) \subset \mathrm{Hom}(X^\bullet, Y^\bullet)$ *is a subgroup. Moreover,*

(1) $\alpha^\bullet \in \mathrm{Hom}^0(X^\bullet, Y^\bullet)$ *implies that* $H^n(\alpha^\bullet) = 0$ *for every* $n \in \mathbb{Z}$.

(2) *For every* $\alpha^\bullet \in \mathrm{Hom}^0(X^\bullet, Y^\bullet)$ *and* $\beta^\bullet \in \mathrm{Hom}(Y^\bullet, Z^\bullet)$, $\gamma^\bullet \in \mathrm{Hom}(W^\bullet, X^\bullet)$, *we have*

$$\beta^\bullet \circ \alpha^\bullet \in \mathrm{Hom}^0(X^\bullet, Z^\bullet) \qquad \textit{and} \qquad \alpha^\bullet \circ \gamma^\bullet \in \mathrm{Hom}^0(W^\bullet, Y^\bullet)$$

(often said in words as "null–homotopic morphisms form an ideal").

Proof. (I am going to drop the superscripts for the easy of reading).

Null–homotopic morphisms form a subgroup. Clearly zero morphism is null–homotopic. Assume that $\alpha, \alpha' \sim 0$ are two null–homotopic morphisms. That is, we have $r^n, s^n : X^n \to Y^{n-1}$ (for every $n \in \mathbb{Z}$) such that $\alpha = dr + rd$ and $\alpha' = ds + sd$. Then:

$$\alpha - \alpha' = d(r - s) + (r - s)d \ ,$$

proving that it is also null–homotopic.

$\alpha \sim 0 \Rightarrow H^n(\alpha) = 0$. Fix $n \in \mathbb{Z}$ and consider the commutative diagram sketched above in the definition of null–homotopic morphisms. That is, $\alpha^n = d^{n-1}s^n + s^{n+1}d^n$. Now, if $x \in \mathrm{Ker}(d_X^n)$, then $\alpha^n(x) = d^{n-1}(s^n(x)) \in \mathrm{Im}(d_Y^{n-1})$, proving that at the level of cohomology, we get $H^n(\alpha) = 0$.

Null–homotopic morphisms form an ideal. Let $\alpha : X \to Y$ be null–homotopic, with homotopy s so that $\alpha = ds + sd$. Let $\beta : Y \to Z$ be arbitrary. We have

$$\beta \circ \alpha = \beta ds + \beta sd = d\beta s + \beta sd = dr + rd \ ,$$

where $r^n = \beta^{n-1} \circ s^n : X^n \to Z^{n-1}$. In the equalities above, we have used the fact that β being a morphism, commutes with d. Hence $\beta\alpha \sim 0$ with homotopy $\beta \circ s$. $\square$