LECTURE 15

(15.0) Complexes and related functors.– Recall that in the last lecture, we narrowed our attention to the abelian category $\mathcal{A} = R$-mod, where R is a unital ring.

- We defined the category of complexes $\mathbb{K}^\bullet(\mathcal{A})$, and verified that it is an abelian category.

- For each $n \in \mathbb{Z}$, we constructed three functors:
$$\mathcal{Z}^n, \mathcal{B}^n, H^n : \mathbb{K}^\bullet(\mathcal{A}) \to \mathcal{A}.$$
Note that for each $X^\bullet \in \mathbb{K}^\bullet(\mathcal{A})$, we get a short exact sequence:
$$0 \to \mathcal{B}^n(X^\bullet) \to \mathcal{Z}^n(X^\bullet) \to H^n(X^\bullet) \to 0.$$

For later use in these notes, we notice that $H^n(X^\bullet)$ can also be defined as follows. Since for every $n \in \mathbb{Z}$, $\mathcal{B}^n(X^\bullet) \subset \mathrm{Ker}(d^n)$, the differential d^n gives rise to a morphism

$$\underline{d^n} : X^n/\mathcal{B}^n(X^\bullet) \to \mathcal{Z}^{n+1}(X^\bullet).$$

$$H^n(X^\bullet) = \mathrm{Ker}\left(\underline{d^n} : X^n/\mathcal{B}^n(X^\bullet) \to \mathcal{Z}^{n+1}(X^\bullet)\right), \tag{1}$$
$$H^{n+1}(X^\bullet) = \mathrm{CoKer}\left(\underline{d^n} : X^n/\mathcal{B}^n(X^\bullet) \to \mathcal{Z}^{n+1}(X^\bullet)\right). \tag{2}$$

(15.1) Theorem of the day.– The main result we are going to prove today is the following theorem.

Theorem. *Let C_1, C_2, C_3 be three complexes* [1] *in $\mathbb{K}^\bullet(\mathcal{A})$ and assume that we are given a short exact sequence:*

$$0 \to C_1^\bullet \xrightarrow{\alpha^\bullet} C_2^\bullet \xrightarrow{\beta^\bullet} C_3^\bullet \to 0.$$

Then, there exists a morphism $\delta^n : H^n(C_3) \to H^{n+1}(C_1)$ with the following properties.

Long exact sequence. *The following sequence is exact.*

$$\cdots H^n(C_1) \xrightarrow{H^n(\alpha)} H^n(C_2) \xrightarrow{H^n(\beta)} H^n(C_3) \xrightarrow{\delta^n} H^{n+1}(C_1) \xrightarrow{H^{n+1}(\alpha)} H^{n+1}(C_2) \xrightarrow{H^{n+1}(\beta)} H^{n+1}(C_3) \quad \cdots$$

Naturality. *If there are two short exact sequences with morphisms forming commutative squares:*

[1] I am omitting the $\bullet$ in the superscript for easy of typing and reading.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_1 & \xrightarrow{\alpha_1} & C_2 & \xrightarrow{\beta_1} & C_3 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \\
0 & \longrightarrow & D_1 & \xrightarrow{\alpha_2} & D_2 & \xrightarrow{\beta_2} & D_3 & \longrightarrow & 0
\end{array}$$

then the following diagram commutes:

$$\begin{array}{ccc}
H^n(C_3) & \xrightarrow{\delta^n_C} & H^{n+1}(C_1) \\
\downarrow{\scriptstyle H^n(f_3)} & & \downarrow{\scriptstyle H^{n+1}(f_1)} \\
H^n(D_3) & \xrightarrow[\delta^n_D]{} & H^{n+1}(D_1)
\end{array}$$

The proof of this theorem relies on the famous "Snake lemma".

(15.2) Snake lemma.– Assume we are given the following commutative diagram of morphisms between left R–modules, where the rows are assumed to be exact (at the middle term).

$$\begin{array}{ccccc}
M_1 & \xrightarrow{u} & M_2 & \xrightarrow{v} & M_3 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
N_1 & \xrightarrow{u'} & N_2 & \xrightarrow{v'} & N_3
\end{array}$$

Using Lemma 14.1, we obtain the following commutative diagram. Note that we are not saying anything about the exactness of the first and last rows.

$$\begin{array}{ccccc}
\mathrm{Ker}(f) & \xrightarrow{u_1} & \mathrm{Ker}(g) & \xrightarrow{v_1} & \mathrm{Ker}(h) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} \\
M_1 & \xrightarrow{u} & M_2 & \xrightarrow{v} & M_3 \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
N_1 & \xrightarrow{u'} & N_2 & \xrightarrow{v'} & N_3 \\
\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle r} \\
\mathrm{CoKer}(f) & \xrightarrow[u_2]{} & \mathrm{CoKer}(g) & \xrightarrow[v_2]{} & \mathrm{CoKer}(h)
\end{array}$$

Here i, j, k are canonical inclusions, and p, q, r are canonical projections. The maps u_1, v_1 and u_2, v_2 are uniquely defined by the universal properties of kernels and cokernels.

Lemma.

(1) *If u is injective, then u_1 is injective. Similarly, if v' is surjective, then v_2 is surjective.*

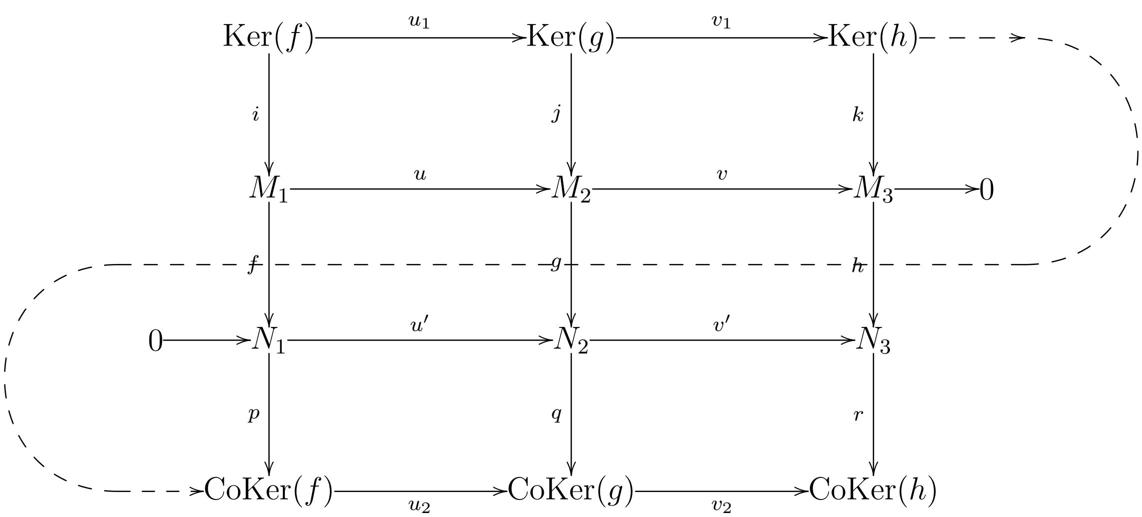
(2) $v_1 \circ u_1 = 0$. *If u' is injective, then the first row* $\mathrm{Ker}(f) \xrightarrow{u_1} \mathrm{Ker}(g) \xrightarrow{v_1} \mathrm{Ker}(h)$ *is exact.*

(3) $v_2 \circ u_2 = 0$. *If v is surjective, then the last row* $\mathrm{CoKer}(f) \xrightarrow{u_2} \mathrm{CoKer}(g) \xrightarrow{v_2} \mathrm{CoKer}(h)$ *is exact.*

(4) *Assume u' is injective and v is surjective. Then, there exists a unique morphism* $\delta : \mathrm{Ker}(h) \to \mathrm{CoKer}(f)$ *making the following sequence exact:*

$$\mathrm{Ker}(f) \xrightarrow{u_1} \mathrm{Ker}(g) \xrightarrow{v_1} \mathrm{Ker}(h) \xrightarrow{\delta} \mathrm{CoKer}(f) \xrightarrow{u_2} \mathrm{CoKer}(g) \xrightarrow{v_2} \mathrm{CoKer}(h).$$

In *all the textbooks* featuring this lemma, the following diagram appears to justify the name *snake.*



Proof. (1): u injective $\Rightarrow u_1$ injective. This is clear since u_1 is merely the restriction of u. In more detail, if $x \in \mathrm{Ker}(u_1)$, then $u(i(x)) = 0$. But both u and i are injective, so $x = 0$. Another argument that avoids picking an element of $\mathrm{Ker}(u_1)$ is as follows. $j \circ u_1 = u \circ i$. As u and i are both injective, so is $u \circ i$ and hence $j \circ u_1$. As j is injective, we conclude that so is u_1. The proof for v', v_2 is similar.

(2). Since $k \circ (v_1 \circ u_1) = v \circ u \circ i = 0$, and k is injective, we get that $v_1 \circ u_1 = 0$. Now assume that u' is injective. We want to show $\mathrm{Ker}(v_1) = \mathrm{Im}(u_1)$. Let $x \in \mathrm{Ker}(v_1)$. Then $v(j(x)) = k(v_1(x)) = 0$, so $j(x) \in \mathrm{Ker}(v) = \mathrm{Im}(u)$. Choose $m_1 \in M_1$ so that $u(m_1) = j(x)$. We only need to check that $m_1 \in \mathrm{Ker}(f)$ to conclude that $x \in \mathrm{Im}(u_1)$. This is where we use the hypothesis that u' is injective. Since $u'(f(m_1)) = g(u(m_1)) = g(j(x)) = 0$, we conclude that $f(m_1) = 0$.

Proof of (3) is absolutely analogous to that of (2) and hence is omitted here.

(4). *Definition of δ.* This is a standard "diagram chase" way to construct the connecting morphism. Let $x \in \mathrm{Ker}(h)$.

(Step 1) Let $m_3 = k(x)$. As v is surjective, we can choose $m_2 \in M_2$ such that $k(x) = v(m_2)$.

(Step 2) Let $n_2 = g(m_2)$. Since $v'(n_2) = v'(g(m_2)) = h(v(m_2)) = h(k(x)) = 0$, by exactness at N_2, and injectivity of u', there is a unique $n_1 \in N_2$ such that $n_2 = u'(n_1)$.

Define $\boxed{\delta(x) = p(n_1)}$. Now we need to show a couple of things.

(i) *Well–defined.* In Step 1, we made a choice, relying on the fact that v is surjective. Assume we choose a different $m_2' \in M_2$, and let n_1', n_2' be the elements corresponding to this choice in the second step. Now $v(m_2 - m_2') = 0$. Then $m_2 - m_2' = u(m)$ for some $m \in M_1$. We get that $g(m_2 - m_2') = u'(f(m))$. By injectivity of u', we conclude that $n_1 - n_1' = f(m)$ and hence $p(n_1) = p(n_1') + p(f(m)) = p(n_1')$.

(ii) *δ is R–linear.* Let $r, r' \in R$, $x, x' \in \text{Ker}(h)$. We need to prove that $\delta(rx + r'x') = r\delta(x) + r'\delta(x')$. In the recipe for defining δ, let m_2, m_2' be the elements chosen at Step 1 for defining $\delta(x)$ and $\delta(x')$ respectively. Note that $v(rm_2 + r'm_2') = rv(m_2) + rv(m_2') = k(rx + r'x')$, meaning we can take $m_2'' = rm_2 + r'm_2'$ in order to define $\delta(rx + r'x')$. The second step goes through and we get the desired result.

(iii) $\delta \circ v_1 = 0$. If $x = v_1(u)$ for some $y \in \text{Ker}(g)$, then we can take $m_2 = j(y)$ in Step 1. This is clear since $v(j(y)) = k(v_1(y)) = k(x)$. This means $n_2 = g(m_2) = g(j(y)) = 0$, giving $n_1 = 0$ and hence $\delta(x) = 0$.

(iv) $\text{Ker}(\delta) \subset \text{Im}(v_1)$. Assume $x \in \text{Ker}(h)$ is such that $\delta(x) = 0$. Keeping the same notations as in the two steps above, this means $p(n_1) = 0$. As p is the cokernel of f, this means $n_1 = f(m)$ for some $m \in M_1$. Consider $m_2 - u(m) \in M_2$.

$$g(m_2 - u(m)) = g(m_2) - u'(f(m)) = n_2 - u'(n_1) = n_2 - n_2 = 0.$$

So there is $y \in \text{Ker}(g)$ such that $j(y) = m_2 - u(m)$. We claim that $x = v_1(y)$. To see this, apply k to $x - v_1(y)$ to get (using $v \circ u = 0$):

$$k(x - v_1(y)) = k(x) - v(j(y)) = m_3 - v(m_2 - u(m)) = m_3 - m_3 = 0.$$

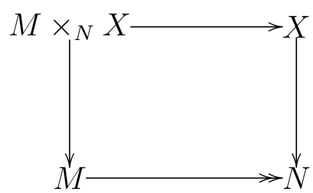
Since k is injective, $x = v_1(y)$.

(v) The exactness at $\text{Ker}(h) \xrightarrow{\delta} \text{CoKer}(f) \xrightarrow{u_2} \text{CoKer}(g)$ is proved similarly. The details are left to the reader. □

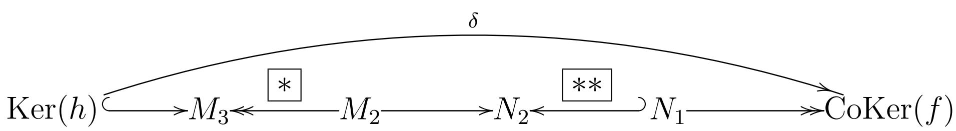
(15.3) Remarks.–
I. There is a proof of the lemma above that works in any abelian category. I did not give that proof, which would have eliminated the need for checking that δ is well–defined and R–linear, at the price of losing the clarity of "diagram chase". If you want to think about that "free of elements" proof, the following tricks from *S. MacLane, Categories for working mathematicians* should help you.

- Replace each $x \in M$ with $X \xrightarrow{x} M$. This very conveniently replaces an expression $f(x)$ by $f \circ x$ for any morphism $f : M \to M'$.

- Choosing an element from the preimage of $n \in N$ under a surjective map $M \twoheadrightarrow N$, gets replaced by taking "pull–back" diagram (see Problem 3 of Mid Term 1):



II. The following summarizes how we defined δ.



- At $\boxed{*}$ we had to use the surjectivity of v and make a choice. We showed that this choice is immaterial at the end.

- At $\boxed{**}$ we had to prove, using exactness at N_2 and injectivity of u', that the element placed at N_2 comes from a unique one from N_1.

(15.4) Proof of Theorem 15.1.– Recall that we have three complexes C_1, C_2, C_3 in $\mathbb{K}^\bullet(\mathcal{A})$ and a short exact sequence $0 \to C_1 \xrightarrow{\alpha} C_2 \xrightarrow{\beta} C_3 \to 0$. Let us fix $n \in \mathbb{Z}$.

(i) Apply Snake lemma (1) and (2) with substitutions: $M_\ell = C_\ell^{n+1}$ and $N_\ell = C_\ell^{n+2}$, ($\ell = 1, 2, 3$) to get the following exact sequence:

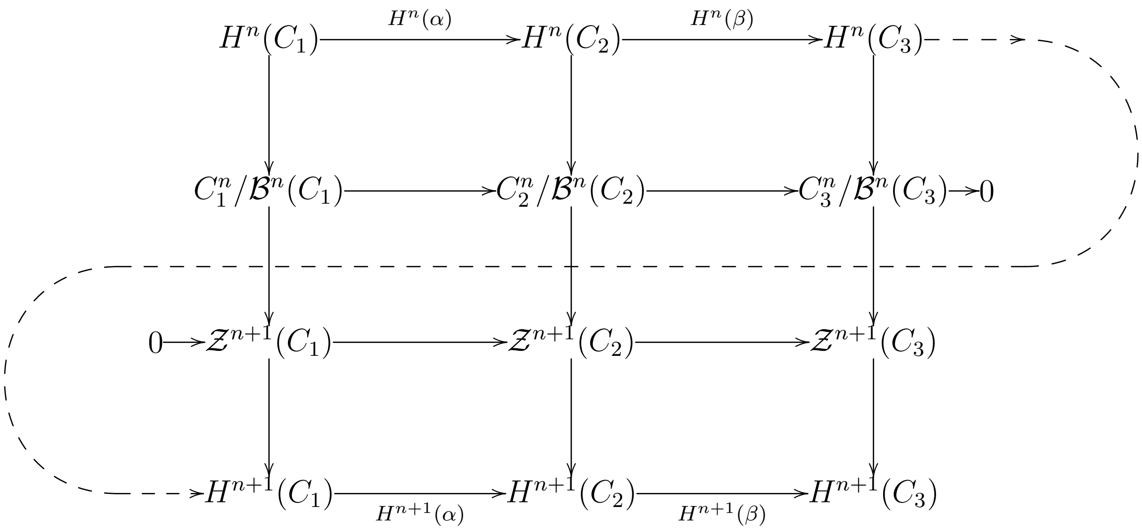
$$0 \to \mathcal{Z}^{n+1}(C_1) \to \mathcal{Z}^{n+1}(C_2) \to \mathcal{Z}^{n+1}(C_3).$$

Remark. Note that this prove that $\mathcal{Z}^n$ is left exact.

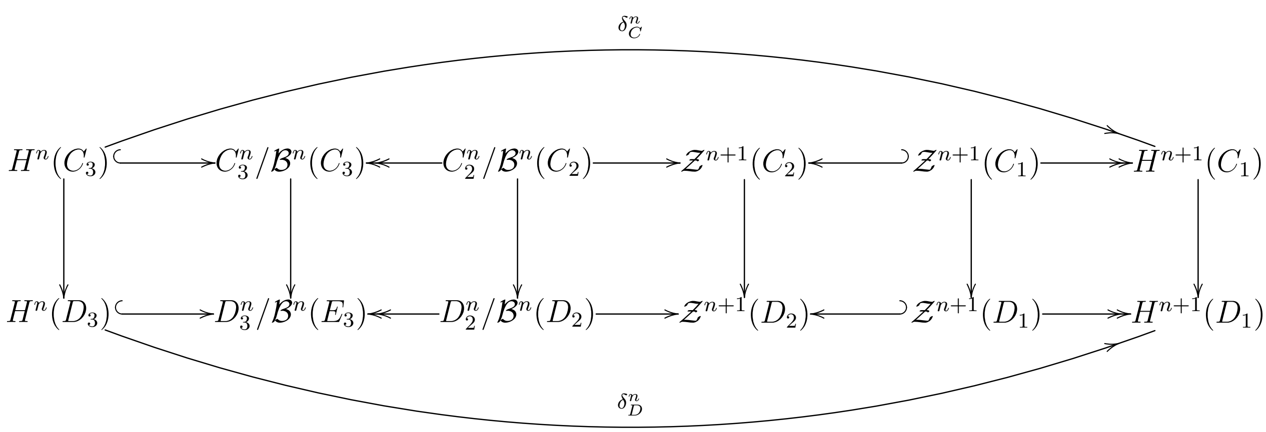
(ii) Again apply the Snake lemma with $M_\ell = C_\ell^{n-1}$ and $N_\ell = C_\ell^n$, ($\ell = 1, 2, 3$), to get the following exact sequence:

$$C_1^n/\mathcal{B}^n(C_1) \to C_2^n/\mathcal{B}^n(C_2) \to C_3^n/\mathcal{B}^n(C_3) \to 0.$$

(iii) Now consider the commutative diagram formed by taking the two exact sequences from the previous part, and use equations (1) and (2) to get:



This finishes the proof of the first part of Theorem 15.1. For the second part (i.e, naturality of δ), we use the diagram that summarized its construction (see §15.3, II above).



Now each square in the picture above commutes by Lemma 14.1, hence so does the outermost one. The naturality of δ follows.