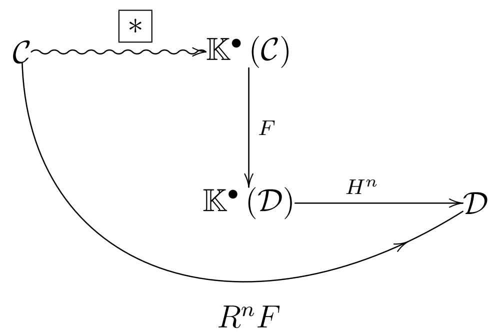
LECTURE 16

(16.0) Road to derived functors.– Recall from Lecture 13, our objective is to follow the path sketched below to get to derived functors. For definiteness, $F : \mathcal{C} \to \mathcal{D}$ is a left exact, covariant functor between two abelian categories $\mathcal{C}$ and $\mathcal{D}$.



- Here $F : \mathbb{K}^\bullet(\mathcal{C}) \to \mathbb{K}^\bullet(\mathcal{D})$ is applied to a complex, term-by-term, and also to the differentials.

$$\left(\cdots X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} \cdots \right) \qquad \mapsto \qquad \left(\cdots F(X^n) \xrightarrow{F(d^n)} F(X^{n+1}) \xrightarrow{F(d^{n+1})} \cdots \right)$$

  Similarly, on morphisms F is applied term-by-term.

- We studied $\{H^n\}_{n\in\mathbb{Z}}$ functors in the last two lectures. They give rise to long exact sequences, and do not distinguish between morphisms homotopic to each other.

- $\boxed{*}$ is not going to be a functor. This is the step where we will have to construct *resolutions* by special objects - injective, projective, free or flat (the meaning of these words will be explained, don't worry). We will also have to prove that the choice of a resolution in this construction becomes immaterial after H^n is applied - namely, *resolutions are unique up to homotopy.* Proofs of these assertions are the focus of this and the next lecture.

As before, $\mathcal{A} = R$-mod is the category of left R–modules. The definitions and results of this lecture work for any abelian category $\mathcal{A}$, and will be stated and proved in that generality. However, for the purposes of this course, you may as well assume that we are talking about modules over R.

(16.1) Injective and projective objects.–

Definition. An object $Q \in \mathcal{A}$ is said to be *injective* if, for every injective morphism $f : Q \to X$, there exists Q', and an isomorphism $g : Q \oplus Q' \xrightarrow{\sim} X$ such that $g \circ \iota_Q = f$, where $\iota_Q : Q \to Q \oplus Q'$ is the canonical inclusion.

Similarly, an object $P \in \mathcal{A}$ is said to be *projective* if, for every surjective morphism $f : Y \to P$, there exists P', and an isomorphism $g : Y \xrightarrow{\sim} P \times P'$ such that $\pi_P \circ g = f$, where $\pi_P : P \times P' \to P$ is the canonical surjection.

Recall (Lecture 8, Lemma 8.3) that finite direct sums and direct products are naturally isomorphic in any additive category. The following characterization of direct sum/product of two objects was used in Lecture 10 (see Claim on page 2). We record here as a lemma for future use, its proof is given in Lecture 10 (page 2).

Lemma. *Let X, X_1, X_2 be three objects in $\mathcal{A}$. Then $X \cong X_1 \oplus X_2$ if, and only if, we have morphisms $f_\ell : X_\ell \to X$ and $g_\ell : X \to X_\ell$ ($\ell = 1, 2$) such that:*

$$g_k \circ f_\ell = \delta_{k,\ell}, \; k, \ell \in \{1, 2\}, \qquad \mathrm{Id}_X = f_1 \circ g_1 + f_2 \circ g_2.$$

(16.2) Idempotents.– The following result is going to be crucial in obtaining different characterizations of injective and projective objects.

Lemma. *Let $X \in \mathcal{A}$ and let $p \in \mathrm{End}_{\mathcal{A}}(X)$ be such that $p \circ p = p$ (such a morphism is called an* idempotent *or* projection*). Then $\mathrm{Ker}(p) \oplus \mathrm{Im}(p) \xrightarrow{\sim} X$.*

Proof. Let $q \in \mathrm{End}_{\mathcal{A}}(X)$ be defined as $\mathrm{Id}_X - p$. Then: $q^2 = (1-p)(1-p) = 1 - 2p + p^2 = 1 - p = q$. Moreover, $pq = p(1-p) = p - p^2 = 0$, and $qp = (1-p)p = p - p^2 = 0$. That is, we have:

$$q \circ q = q, \qquad \text{and} \qquad p \circ q = 0 = q \circ p.$$

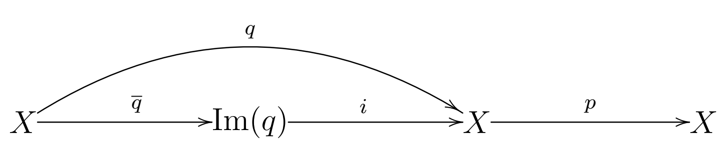
Claim. $\mathrm{Im}(q) = \mathrm{Ker}(p)$ and $\mathrm{Ker}(q) = \mathrm{Im}(p)$.

Let us assume this for now, and see how the lemma follows. Let us denote by $X_1 = \mathrm{Ker}(p) \xrightarrow{i} X$ the canonical inclusion. Moreover, $X \xrightarrow{\overline{q}} X_1 = \mathrm{Im}(q)$ is the natural surjection which factors $q = i \circ \overline{q}$. Similarly,

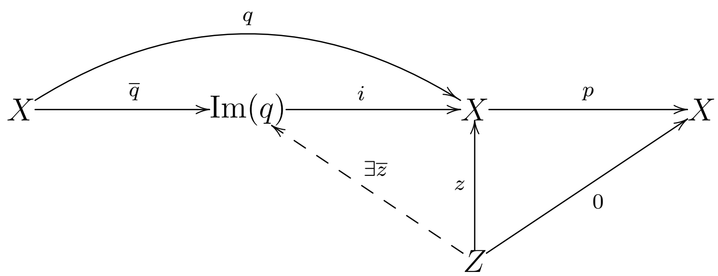
$$X_2 = \mathrm{Ker}(q) \xrightarrow{j} X \xrightarrow{\overline{p}} X_2 = \mathrm{Im}(p).$$

From the properties of p, q listed above, it is easy to see that this 4-tuple of morphisms satisfies the equations listed in Lemma 16.1 above, hence identifies X as a direct sum (or product) of X_1 and X_2.

Proof of the claim. Let us prove this in a more categorical framework. Consider the following diagram:

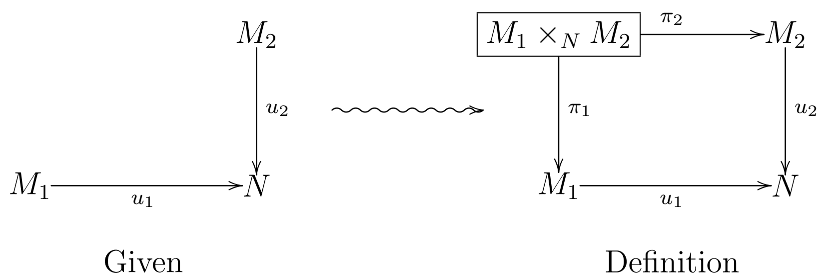


The composition $p \circ i \circ \overline{q} = p \circ q = 0$. As $\overline{q}$ is surjective, we conclude that $p \circ i = 0$. Now to prove that $\mathrm{Im}(q) \xrightarrow{i} X$ is the kernel of p, we have to show that for every morphism $z : Z \to X$ such that $p \circ z = 0$, $z = i \circ \overline{z}$ for a unique $\overline{z}$ (uniqueness is clear since i is injective, we only have to check it exists).



Since $p \circ z = 0$ we get $z = z - p \circ z = (1 - p) \circ z = q \circ z$. Hence $z = i \circ (\overline{q} \circ z) = i \circ \overline{z}$ as we wanted. The proof of $\mathrm{Im}(p) = \mathrm{Ker}(q)$ is analogous and therefore omitted. $\square$

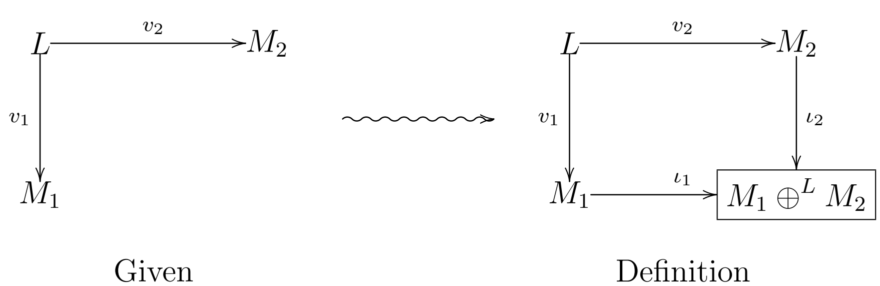
(16.3) Pull–backs and push–forwards.– (See Homework 5, Problems 3 and 4). Recall that pull–back diagram is summarized as:



Concretely $M_1 \times_N M_2$ is defined as the kernel of $M_1 \times M_2 \to N$, with π_1 and π_1 being the following two compositions:

$$M_1 \times_N M_2 \hookrightarrow M_1 \times M_2 \to M_1, \qquad \text{and} \qquad M_1 \times_N M_2 \hookrightarrow M_1 \times M_2 \to M_2.$$

Similarly, the following diagram defines push–forward:



Again, concretely $M_1 \oplus^L M_2$ is the cokernel of $L \to M_1 \oplus M_2$, with morphisms ι_1 and ι_2 given as compositions of natural inclusions and surjections:

$$M_1 \to M_1 \oplus M_2 \twoheadrightarrow M_1 \oplus^L M_2, \qquad \text{and} \qquad M_2 \to M_1 \oplus M_2 \twoheadrightarrow M_1 \oplus^L M_2.$$

The proof of the following proposition is left as part of the homework.

Proposition.

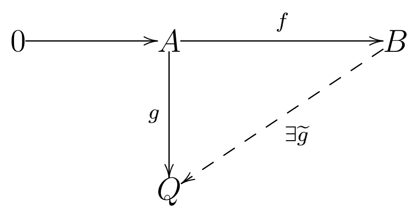
(1) *If* u_1 *is surjective, then so is* π_2.

(2) *If* v_2 *is injective, then so is* ι_1.

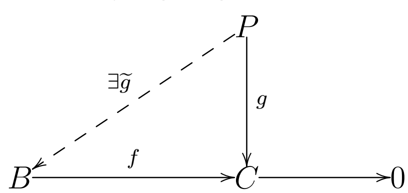
(16.4) Characterizations of injective and projective objects.– Now we are ready to state and prove various different ways injective and projective objects can be defined.

Theorem.

- *Let* $Q \in \mathcal{A}$. *Then the following conditions on* Q *are equivalent.*
  (1) Q *is injective.*
  (2) *For any injective morphism* $f : A \hookrightarrow B$, *and arbitrary* $g : A \to Q$, *there exists a lift* $\widetilde{g} : B \to Q$ *such that* $\widetilde{g} \circ f = g$.



  (3) $\mathrm{Hom}_{\mathcal{A}}(-, Q)$ *is exact.*
- *Let* $P \in \mathcal{A}$. *Then the following conditions on* P *are equivalent.*
  (1) P *is projective.*
  (2) *For any surjective morphism* $f : B \twoheadrightarrow C$, *and arbitrary* $g : P \to C$, *there exists a lift* $\widetilde{g} : P \to B$ *such that* $f \circ \widetilde{g} = g$.

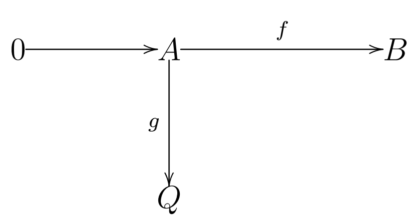


  (3) $\mathrm{Hom}_{\mathcal{A}}(P, -)$ *is exact.*

We will prove this theorem for injective objects only. The proof for projective objects is absolutely analogous and is left to the reader.

(16.5) Proof of Theorem 16.4: injective case.– Let $Q \in \mathcal{A}$.

(1) $\Rightarrow$ (2). Assume that Q is injective, and we have the diagram:



Let $\widetilde{Q} = Q \oplus^A B$ be the push–forward (see §16.3 above). As claimed in Proposition 16.3 (2), the natural map $Q \to \widetilde{Q}$ is injective. By definition, we have an object Q' and an isomorphism $Q \oplus Q' \xrightarrow{\sim} \widetilde{Q}$.

$$\begin{array}{ccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \overline{g}} \\ 0 & \longrightarrow & Q & \xrightarrow[i]{} & Q \oplus Q' = \widetilde{Q} \end{array}$$

Define $\widetilde{g} = p \circ \overline{g}$. Here $p : Q \oplus Q' = Q \times Q' \to Q$ is the natural surjection. Thus,

$$g = p \circ i \circ g = p \circ \overline{g} \circ f = \widetilde{g} \circ f,$$

as claimed.

(2) $\Rightarrow$ (3). Now we assume (2), and prove that $\mathrm{Hom}(-, Q)$ is exact. Since Hom functors are always left exact, we only need to show that it is right exact. That is, given a short exact sequence:

$$0 \to A \xrightarrow{f} B \to C \to 0 \ ,$$

the natural map $\mathrm{Hom}(B, Q) \to \mathrm{Hom}(A, Q)$ is surjective. That is, any morphism $g : A \to Q$ lifts to a morphism $\widetilde{g} : B \to Q$. This is exact what condition (2) says, and we are done. Note that this argument in fact shows that (2) and (3) are equivalent.

(3) $\Rightarrow$ (1). Assume that $Q \in \mathcal{A}$ is such that $\mathrm{Hom}(-, Q)$ is exact. Let $f : Q \to X$ be an injective morphism. We need to prove that there exists Q' and an isomorphism $h : Q \oplus Q' \xrightarrow{\sim} X$ such that $h \circ i = f$, where $i : Q \to Q \oplus Q'$ is the natural inclusion.

We use (3) to conclude that $- \circ f : \mathrm{Hom}(X, Q) \to \mathrm{Hom}(Q, Q)$ is a surjective map. Let $g : X \to Q$ be a morphism so that $g \circ f = \mathrm{Id}_Q$. Let $p : X \to X$ be the composition $p = f \circ g$. Then $p \circ p = f \circ g \circ f \circ g = f \circ g = p$. Using Lemma 16.2 we conclude that $X \cong \mathrm{Ker}(p) \oplus \mathrm{Im}(p)$. Note that $\mathrm{Im}(p) = \mathrm{Im}(f \circ g) \cong Q$, since g is surjective and f is injective. Hence Q is injective.

(16.6) Remarks.–
I. The following is a defining property whose proof follows along the lines of Theorem 16.4 above. It is left as an exercise.

Proposition. *$M \in \mathcal{A}$ is injective (resp. projective) if, and only if every short exact sequence $0 \to M \to Y \to Z \to 0$ (resp. $0 \to X \to Y \to M \to 0$) splits*[1].

II. We have not yet proved whether injective/projective objects exist. The standard examples of these, which can be easily verified using Theorem 16.4 (we will go over these in more detail next week), are following. (i) $\mathbb{Q} \in \mathbb{Z}$-mod $= \mathbf{Ab}$ is an injective $\mathbb{Z}$–module. (ii) For any indexing set I, let $R^{(I)}$ be direct sum of I–many copies of R (viewed as a left module over itself). Then $R^{(I)} \in R$-mod is projective (it is in fact *free*). (iii) Over a field, every module is both injective and projective.

[1]See Homework 5, Problem 8 for what it means.