LECTURE 17

(17.0) Injective and projective objects.– Recall that in the last lecture we defined injective and projective objects in $\mathcal{A} = R$-mod. Our definition was phrased for any abelian category, and is given as follows.

Injective objects. An object $Q \in \mathcal{A}$ is called an injective object, if it satisfies (one of the) following equivalent conditions.

- For any injective morphism $f : Q \hookrightarrow \widetilde{Q}$ in $\mathcal{A}$, there exists $Q' \in \mathcal{A}$ and an isomorphism $g : Q \oplus Q' \xrightarrow{\sim} \widetilde{Q}$ such that $g \circ i = f$. Here, $i : Q \hookrightarrow Q \oplus Q'$ is the natural inclusion.

- For any injective morphism $f : A \to B$, and an arbitrary morphism $g : A \to Q$, there exists a lift $\widetilde{g} : B \to Q$ such that $\widetilde{g} \circ f = g$.

- $\mathrm{Hom}(-, Q)$ is exact.

- Every short exact sequence $0 \to Q \to Y \to Z \to 0$ splits.

Projective objects. An object $P \in \mathcal{A}$ is called a projective object, if it satisfies (one of the) following equivalent conditions.

- For any surjective morphism $f : \widetilde{P} \twoheadrightarrow P$ in $\mathcal{A}$, there exists $P' \in \mathcal{A}$ and an isomorphism $g : \widetilde{P} \xrightarrow{\sim} P \times P'$ such that $\pi \circ g = f$. Here, $\pi : P \times P' \to P$ is the natural surjection.

- For any surjective morphism $f : B \to C$, and an arbitrary morphism $g : P \to C$, there exists a lift $\widetilde{g} : P \to B$ such that $f \circ \widetilde{g} = g$.

- $\mathrm{Hom}(P, -)$ is exact.

- Every short exact sequence $0 \to X \to Y \to P \to 0$ splits.

Today we are going to discuss injective and project *resolutions*.

(17.1) Resolutions.– Let $M \in \mathcal{A}$.

Definition. An *injective resolution* of M is a *cochain* complex in $\mathbb{K}^\bullet(\mathcal{A})$,

$$Q^\bullet \quad : \qquad 0 \to Q^0 \xrightarrow{d^0} Q^1 \xrightarrow{d^1} Q^2 \xrightarrow{d^2} \cdots \ ,$$

which is exact at each Q^n $(n \geq 1)$, and $\mathrm{Ker}(d^0) \cong M$. Here, $Q^{-\ell} = 0$ for every $\ell \geq 1$.

A *projective resolution* of M is a *chain* complex in $\mathbb{K}_\bullet(\mathcal{A})$,

$$P_\bullet \;: \qquad \cdots \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \to 0,$$

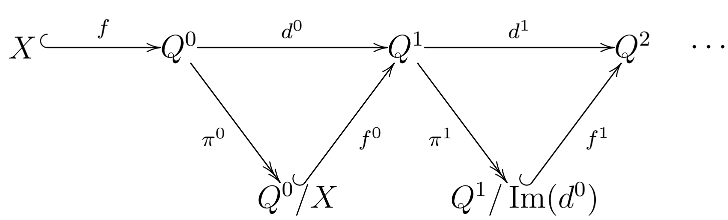
which is exact at each P_n ($n \geq 1$), and $\mathrm{CoKer}(d_0) \cong M$. Here $P_{-\ell} = 0$ for every $\ell \geq 1$.

The existence of such resolutions is a technical requirement imposed on the category in question.

(17.2) Enough injectives/projectives.– Let $\mathcal{C}$ be an arbitrary abelian category. We say that $\mathcal{C}$ has *enough injectives* (resp. *enough projectives*) if for every $X \in \mathcal{C}$, there exists an injective morphism $X \hookrightarrow Q$ (resp. surjective morphism $P \twoheadrightarrow X$) where Q is injective (resp. P is projective).

Building injective resolutions. If $\mathcal{C}$ has enough injectives, then for every $X \in \mathcal{C}$, an injective resolution $Q^\bullet$ of X can be constructed as follows.

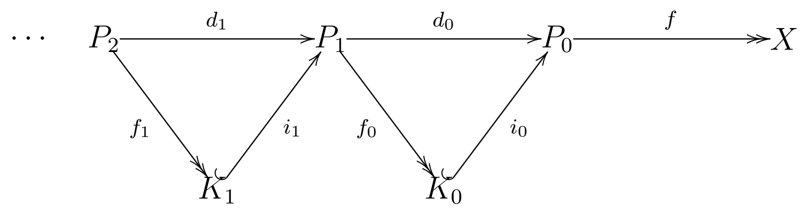
- Start from an injective morphism $f : X \hookrightarrow Q^0$, where Q^0 is injective.
- Let $\pi^0 : Q^0 \twoheadrightarrow X^0 = \mathrm{CoKer}(f) = Q^0/X$. Find an injective morphism $f^0 : Q^0/X \hookrightarrow Q^1$ to an injective object Q^1. Define $d^0 = f^0 \circ \pi^0$.
- Repeat the previous step indefinitely...



- $Q^\bullet \;:= \qquad 0 \to Q^0 \xrightarrow{d^0} Q^1 \xrightarrow{d^1} Q^2 \to \cdots$ It is clear from its construction that the complex is exact at each spot, except $\mathrm{Ker}(d^0) = X$.

Building projective resolutions. This construction is entirely analogous to the one given above. Namely,

- Start from a surjection from a projective object $f : P_0 \twoheadrightarrow X$.
- Let $i_0 : K_0 = \mathrm{Ker}(f) \hookrightarrow P_0$. Find a surjection from a projective object $f_0 : P_1 \twoheadrightarrow K_0$. Define $d_0 = i_0 \circ f_0 : P_1 \to P_0$.
- Repeat.



- $P_\bullet :=\quad \cdots \to P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \to 0$. Again, this is an exact (chain) complex at each spot, except $\mathrm{CoKer}(d_0) = X$.

We record these constructions as a proposition.

Proposition. *Assuming an abelian category $\mathcal{C}$ has enough injectives (resp. projectives), every $X \in \mathcal{C}$ admits an injective (resp. projective) resolution.*

Remarks.

(1) Injective/projective resolutions are not unique, but only up to homotopy, as we will prove below.

(2) In practice, projective resolutions are easier to work with, than injective ones.

(3) We will have to prove later that $\mathcal{A} = R$-mod has enough injectives and projectives. The projective case is easy, but it will take some work to show that there are enough injective R–modules.

(17.3) Uniqueness (up to homotopy) of resolutions.– Let $M, N \in \mathcal{A}$ and let $I^\bullet$ and $J^\bullet$ be injective resolutions of M and N respectively. Recall that this means the following two sequences are exact:

$$0 \longrightarrow M \xrightarrow{i} I^0 \xrightarrow{d_I^0} I^1 \xrightarrow{d_I^1} I^2 \xrightarrow{d_I^2} \cdots$$

$$0 \longrightarrow N \xrightarrow{j} J^0 \xrightarrow{d_J^0} J^1 \xrightarrow{d_J^1} J^2 \xrightarrow{d_J^2} \cdots$$

Theorem. *Given a morphism $f : M \to N$, there exists a lift $f^\bullet : I^\bullet \to J^\bullet$. That is, there are morphisms $\{f^n : I^n \to J^n\}_{n\geq 0}$ such that all the squares in the following diagram are commutative:*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \xrightarrow{i} & I^0 & \xrightarrow{d_I^0} & I^1 & \xrightarrow{d_I^1} & I^2 & \xrightarrow{d_I^2} & \cdots \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \downarrow{\scriptstyle f^2} & & \\
0 & \longrightarrow & N & \xrightarrow[j]{} & J^0 & \xrightarrow[d_J^0]{} & J^1 & \xrightarrow[d_J^1]{} & J^2 & \xrightarrow[d_J^2]{} & \cdots
\end{array}$$

Moreover, if $f^\bullet$ and $g^\bullet$ are two lifts of f, then $f^\bullet$ is homotopic to $g^\bullet$.

We will prove this theorem in the next two sections. For now, we record its important consequence.

Corollary. *Injective resolutions are unique up to homotopy.*

Proof. Let $M \in \mathcal{A}$, and let $I^\bullet$ and $J^\bullet$ be two injective resolutions. Take $M = N$ and $f = \mathrm{Id}_M$ in the previous theorem, to get $\alpha : I^\bullet \to J^\bullet$ and $\beta : J^\bullet \to I^\bullet$. Then $\alpha \circ \beta : J^\bullet \to J^\bullet$ and $\beta \circ \alpha : I^\bullet \to I^\bullet$ are both lifts of $\mathrm{Id}_M : M \to M$. But so are identity morphisms $\mathrm{Id}_{J^\bullet}$ and $\mathrm{Id}_{I^\bullet}$. By uniqueness part of the theorem, we conclude that $\alpha \circ \beta \sim \mathrm{Id}_{J^\bullet}$ and $\beta \circ \alpha \sim \mathrm{Id}_{I^\bullet}$. Hence $I^\bullet$ and $J^\bullet$ are isomorphic (up to homotopy). $\square$

(17.4) Proof of Theorem 17.3: existence of lifts.– We begin by constructing $f^0 : I^0 \to J^0$. For this we need to use the fact that i is an injective morphism and J^0 is an injective object:

$$\begin{array}{ccccc}
0 & \longrightarrow & M & \xrightarrow{i} & I^0 \\
 & & \downarrow{\scriptstyle j\circ f} & \swarrow{\scriptstyle \exists f^0} & \\
 & & J^0 & &
\end{array}$$

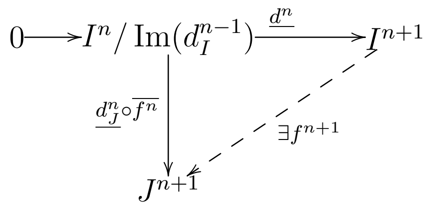
Assume that we have constructed $\{f^\ell : I^\ell \to J^\ell\}_{0 \le \ell \le n}$. Let us construct f^{n+1}. For this, we use the induced morphism at the level of cokernels:

$$\begin{array}{ccccc}
I^{n-1} & \xrightarrow{d_I^{n-1}} & I^n & \longrightarrow & I^n/\operatorname{Im}(d_I^{n-1}) \\
\downarrow{\scriptstyle f^{n-1}} & & \downarrow{\scriptstyle f^n} & & \downarrow{\scriptstyle \overline{f^n}} \\
J^{n-1} & \xrightarrow[d_J^{n-1}]{} & J^n & \longrightarrow & J^n/\operatorname{Im}(d_J^{n-1})
\end{array}$$

As $\operatorname{Im}(d_I^{n-1}) = \operatorname{Ker}(d_I^n)$, we get an injective morphism $\underline{d_I^n} : I^n/\operatorname{Im}(d_I^{n-1}) \hookrightarrow I^{n+1}$ coming from d^n. Similarly we have a morphism, since $\operatorname{Im}(d_J^{n-1}) \subset \operatorname{Ker}(d_J^n)$:

$$\underline{d_J^n} : J^n/\operatorname{Im}(d_J^{n-1}) \to J^{n+1}.$$

Now we can use the fact that J^{n+1} is injective, to get f^{n+1} as:



Remark. Note that in the proof above, we only used the following assumptions:

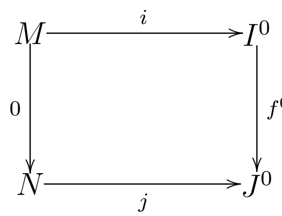
- $0 \to M \to I^0 \to I^1 \to \cdots$ is exact. We did not use the hypothesis that I^ℓ is injective ($\ell \geq 0$).

- $0 \to N \to J^0 \to J^1 \to \cdots$ is a complex, where each J^ℓ is injective. We did not use exactness of this complex.

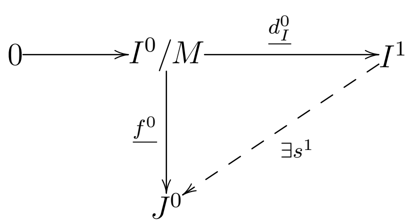
(17.5) Uniqueness of lift.– Note that it suffices to show that if $f^\bullet : I^\bullet \to J^\bullet$ is a lift of the zero morphism $M \xrightarrow{0} N$, then $f^\bullet$ is null–homotopic. That is, there exists $\{s^n : I^n \to J^{n-1}\}_{n\geq 0}$ (with $s^0 = 0$ since $J^{-1} = 0$), such that

$$f^n = d_J^{n-1} \circ s^n + s^{n+1} \circ d_I^n : I^n \to J^n,\ \forall\, n \geq 0.$$

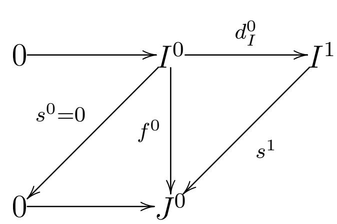
Let us begin by exhibiting how $s^1 : I^1 \to J^0$ shows up. For this, we notice that from the commutative diagram



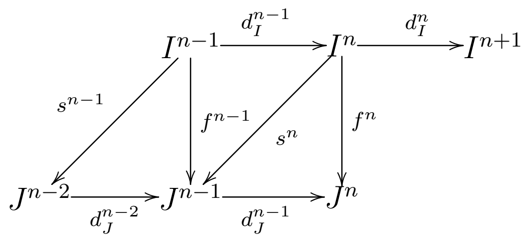
we get that $M \subset \operatorname{Ker}(f^0)$. Thus, f^0 factors through $\underline{f^0} : I^0/M \to J^0$. Moreover, $M = \operatorname{Im}(i) = \operatorname{Ker}(d_I^0)$ gives an injective morphism $\underline{d_I^0} : I^0/M \hookrightarrow I^1$. Now we are in the position to use the fact that J^0 is injective to get $s^1 : I^1 \to J^0$ as follows.



It is easy to see that we get $f^0 = s^1 \circ d_I^0$ so that we have built the required homotopy up to I^1:



Assuming we have successfully built homotopies $\{s^0, \dots, s^n\}$, where $n \geq 1$. We proceed with the construction of s^{n+1}. For this, we have to focus on the following part of the picture:



Define $g = f^n - d_J^{n-1} \circ s^n : I^n \to J^n$. The following calculation shows that $g \circ d_I^{n-1} = 0$:

$$g \circ d_I^{n-1} = f^n d_I^{n-1} - d_J^{n-1} s^n d_I^{n-1} = d_J^{n-1}(f^{n-1} - s^n d_I^{n-1}) = d_J^{n-1} d_J^{n-2} s^{n-1} = 0.$$

Here, we have used the following facts:

- $f^\bullet$ is a morphism of complexes, so $f^n d_I^{n-1} = d_J^{n-1} f^{n-1}$.
- $f^{n-1} = d_J^{n-2} s^{n-1} + s^n d_I^{n-1}$.
- $d_J^{n-1} d_J^{n-2} = 0$.

Thus, we conclude that $\mathrm{Im}(d_I^{n-1}) = \mathrm{Ker}(d_I^n) \subset \mathrm{Ker}(g)$. This allows us to factor g through $\underline{g} : I^n / \mathrm{Ker}(d^n) \to J^n$. Again we rely on injectivity of I^{n+1} in the following diagram to get s^{n+1}: