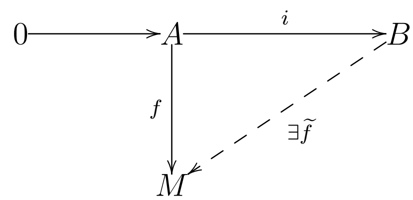
LECTURE 18

(18.0) Theorem of the day.– This lecture is aimed at studying injective modules over R. For simplicity, we will assume that R is a unital *commutative* ring. Recall the definition of an injective module from Theorem 16.4. For today, the property that is going to play the central role is the following.

An R–module M is injective if, and only if for every R–module B, a submodule $A \subset B$, and an R–linear map $f : A \to M$, there exists $\widetilde{f} : B \to M$ such that $\widetilde{f}|_A = f$.



We will show today that

$$\boxed{R\text{-mod has enough injectives}}$$

Recall this means that every module embeds into some injective module. Our proof goes through a few steps:

(Step 1) Obtain a more working criterion for checking something is injective (Baer's criterion in §18.1).

(Step 2) $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in **Ab** (see §18.2 and §18.3 for what it means).

(Step 3) If $Q \in \mathbf{Ab}$ is an injective cogenerator, then $I_Q = \mathrm{Hom}_{\mathbb{Z}}(R, Q) \in R$-mod is an injective cogenerator (Proposition 18.3).

(Step 4) Given an injective cogenerator $I \in R$-mod, every M embeds into $\displaystyle\prod_{\varphi \in \mathrm{Hom}_R(M,I)} I$, the latter being injective (Theorem 18.4).

Thus, using the results from Lecture 17, we have the following theorem.

Theorem. *For every $M \in R$-mod, there exists a unique (up to homotopy) injective resolution $I^\bullet_M \in \mathbb{K}^\bullet(R\text{-mod})$ of M. This assignment $M \mapsto I^\bullet_M$ is "functorial up to homotopy".*

By the last statment, we mean that for every morphism $M \to N$, we have a unique (up to homotopy) lift $I_M^\bullet \to I_N^\bullet$.

(18.1) Baer's criterion for injectivity.– The following lemma gives a more manageable condition to verify that a module is injective. Recall that an ideal $\mathfrak{a} \subset R$ is a subgroup (under addition) such that $ra \in \mathfrak{a}$, $\forall r \in R, a \in \mathfrak{a}$. In other words, $\mathfrak{a} \subset R$ is an R–submodule.

Lemma. *An R–module M is injective if, and only if for every* ideal $\mathfrak{a} \subset R$ *and an R–linear map $f : \mathfrak{a} \to M$, there exists an R–linear $\widetilde{f} : R \to M$ such that $\widetilde{f}|_{\mathfrak{a}} = f$.*

Remark. Since $\mathrm{Hom}_R(R, M) \xrightarrow{\sim} M$ via $\xi \mapsto \xi(1)$, the condition of the lemma above can be equivalently written as follows.

Given an ideal $\mathfrak{a} \subset R$ and an R–linear $f : \mathfrak{a} \to M$, there exists $m \in M$ such that $f(a) = am$ for every $a \in \mathfrak{a}$.

Proof. Since an ideal $\mathfrak{a} \subset R$ is same as an R–submodule of R, the condition stated in the lemma above is clearly necessary. We will now show that it is also sufficient.

So, let B be an R–module, and $A \subset B$ be a submodule. Assume we are given an R–linear map $f : A \to M$. Consider the following partially ordered set of R–submodules of B:

$$\mathcal{P} = \{(A_1, f_1) : A \subset A_1 \subset B \text{ submodule, } f_1 : A_1 \to M \text{ and } f_1|_A = f\}.$$

The partial order on $\mathcal{P}$ is by inclusion. Namely $(A_1, f_1) \leq (A_2, f_2)$ means $A_1 \subset A_2$ and $f_2|_{A_1} = f_1$. Moreover $(A, f) \in \mathcal{P}$, so it is non–empty.

Claim. Every chain in $\mathcal{P}$ has a supremum.

Let us assume this claim for now and proceed with the proof of the lemma. The claim allows us to apply Zorn's lemma and conclude the existence of a maximal $(\widetilde{A}, \widetilde{f}) \in \mathcal{P}$. Now we show that $\widetilde{A} = B$.

Assume on the contrary that $\widetilde{A} \subsetneq B$. Let $x \in B \setminus \widetilde{A}$. Define $\mathfrak{a} = \{r \in R : rx \in \widetilde{A}\} \subset R$. It is a routine exercise (left to the reader) to verify that $\mathfrak{a}$ is an ideal. Consider the R–linear map $g : \mathfrak{a} \to M$ given by

$$g : \mathfrak{a} \to M, \qquad g(a) := \widetilde{f}(ax).$$

This map is R–linear: for every $r_1, r_2 \in R$ and $a_1, a_2 \in \mathfrak{a}$, we have

$$\begin{aligned} g(r_1a_1 + r_2a_2) &= \widetilde{f}((r_1a_1 + r_2a_2)x) = \widetilde{f}(r_1(a_1x)) + \widetilde{f}(r_2(a_2x)) \\ &= r_1\widetilde{f}(a_1x) + r_2\widetilde{f}(a_2x) = r_1g(a_1) + r_2g(a_2). \end{aligned}$$

Now by the hypothesis of the lemma (see the remark after the statement), there exists $m \in M$ such that $g(r) = \widetilde{f}(rx) = rm$ for every $r \in \mathfrak{a}$. Consider $C = \widetilde{A} + Rx \subset M$ and let $h : C \to M$ be given by:

$$h(a) = \widetilde{f}(a),\ \forall a \in \widetilde{A} \qquad \text{and} \qquad h(rx) = rm,\ \forall r \in R.$$

h is well–defined. Note that every element of C can be written as $a + rx$ for some $a \in \widetilde{A}$ and $r \in R$, though this expression need not be unique (that is, the sum $\widetilde{A} + Rx$ need not be direct). In order to make sure that the formula of h is unambiguous, we need to verify that if $rx = a \in \widetilde{A}$ for some $r \in R$, then the two ways to apply h are the same. That is, $\widetilde{f}(a) = rm$. This is true, because $rx \in \widetilde{A}$ implies $r \in \mathfrak{a}$ and by definition of $m \in M$, we have $\widetilde{f}(rx) = g(r) = rm$.

Hence $(C, h) \in \mathcal{P}$ is strictly larger than $\widetilde{A}$, contradicting its maximality. The lemma follows, modulo the claim, which we prove now.

Proof of the claim. Consider a chain in $\mathcal{P}$

$$(A_1, f_1) \leq (A_2, f_2) \leq \cdots$$

and define $A' = \bigcup_{j \geq 1} A_j$. It is easy to see that $A' \subset B$ is a submodule. Define $f' : A' \to M$ as follows:

$$\text{For } a \in A' \text{ choose } j \text{ such that } a \in A_j. \qquad f'(a) := f_j(a).$$

Note that this is well–defined, since if $a \in A_k$ and assuming without loss of generality that $j \leq k$, then $f_k(a) = f_j(a)$ by definition of $\leq$ on $\mathcal{P}$. This $(A', f') \in \mathcal{P}$ is the required supremum and we are done.

It is worth noting that (A', f') is exactly the direct limit:

$$A' = \varinjlim_{j \geq 1} A_j \qquad \text{and} \qquad f' = \varinjlim_{j \geq 1} f_j.$$

□

(18.2) Examples.– Let us use Baer's criterion to prove that $\mathbb{Q} \in \mathbf{Ab}$ is an injective abelian group. We have to show that any map from an ideal in $\mathbb{Z}$ to $\mathbb{Q}$ extends to one $\mathbb{Z} \to \mathbb{Q}$. Recall that ideals in $\mathbb{Z}$ are $\{(n) = n\mathbb{Z}\}_{n \geq 0}$. Assume that $n > 0$ (the case of the zero ideal is always trivial and hence needs no further consideration). Let $f : (n) \to \mathbb{Q}$ be an arbitrary group homomorphism. Now $f(n) = x \in \mathbb{Q}$. Since we can divide by n, define $\widetilde{f} : \mathbb{Z} \to \mathbb{Q}$ by $\widetilde{f}(1) = \dfrac{x}{n}$. It is easy to see that $\widetilde{f}|_{(n)} = f$ and we are done.

Another very important example of an injective abelian group is $\mathbb{Q}/\mathbb{Z}$.

Lemma. *$\mathbb{Q}/\mathbb{Z} \in \mathbf{Ab}$ is an injective abelian group. Moreover, for any abelian group $A \in \mathbf{Ab}$ and $0 \neq a \in A$, there exists a group homomorphism $\psi : A \to \mathbb{Q}/\mathbb{Z}$ such that $\psi(a) \neq 0$.*

Remark. Note that the second assertion of the lemma is false for $\mathbb{Q}$. For instance, take $A = \mathbb{Z}/N\mathbb{Z}$ where $N \geq 2$ is an arbitrary integer. There are no non–zero group homomorphisms from $\mathbb{Z}/N\mathbb{Z} \to \mathbb{Q}$ since $\mathbb{Q}$ has no torsion element. By contrast, every element of $\mathbb{Q}/\mathbb{Z}$ is torsion, and for every $N \in \mathbb{Z}_{\geq 2}$ there are N–torsion elements in $\mathbb{Q}/\mathbb{Z}$, which is what we need to prove this lemma.

Proof. We leave the proof of injectivity of $\mathbb{Q}/\mathbb{Z}$ to the reader (it is exactly the same as the one for $\mathbb{Q}$ given above). Let us prove the second statement. Let $A \in \mathbf{Ab}$ and $0 \neq a \in A$. Take the subgroup generated by a, $A_1 \subset A$. We have two options (depending on whether there exists $N \in \mathbb{Z}_{\geq 2}$ such that $Na = 0$ or not):

$$A_1 \cong \mathbb{Z}, \qquad \text{or} \qquad A_1 \cong \mathbb{Z}/N\mathbb{Z}.$$

In the first case, we can send $a \in A_1$ to any non–zero element to get $f : A_1 \to \mathbb{Q}/\mathbb{Z}$, which then extends to $\psi : A \to \mathbb{Q}/\mathbb{Z}$, since the latter is injective. Then $\psi(a) = f(a) \neq 0$.

In the second case, let $f(a) := \frac{1}{N} \in \mathbb{Q}/\mathbb{Z}$. This is a group homomorphism and its extension $\psi : A \to \mathbb{Q}/\mathbb{Z}$ satisfies the requirement of the lemma. □

(18.3) Injective cogenerators.– An R–module E is said to be a *cogenerator* if for every $M \in R$-mod and $0 \neq m \in M$, there exists an R–linear map $\psi : M \to E$ such that $\psi(m) \neq 0$.

Lemma 18.2 says that $\mathbb{Q}/\mathbb{Z} \in \mathbf{Ab}$ is an injective cogenerator in the category $\mathbf{Ab} = \mathbb{Z}$-mod. Now we will prove that we can use it to construct an injective cogenerator in R-mod, for any R.

Proposition. *Injective cogenerators exist in R-mod. More precisely, let $Q \in \mathbf{Ab}$ be any injective cogenerator in $\mathbf{Ab}$. Define an R–module I_Q as:*

$$I_Q := \mathrm{Hom}_{\mathbb{Z}}(R, Q), \qquad r \in R, \xi \in I_Q \rightsquigarrow (r \cdot \xi)(x) := \xi(rx).$$

Then I_Q is an injective cogenerator in R-mod.

Proof. We need to prove two things: (i) I_Q is an injective R–module, and (ii) I_Q is a cogenerator.

I_Q is injective. Recall that we need to show that for any injective R–linear map $i : A \hookrightarrow B$, the induced homomorphism on Hom's:

$$- \circ i : \mathrm{Hom}_R(B, I_Q) \to \mathrm{Hom}_R(A, I_Q) \text{ is surjective.}$$

Now we use Problem 11 of Homework 6, to conclude that

$$\beta : \mathrm{Hom}_R(M, I_Q) = \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, Q)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R R, Q) = \mathrm{Hom}_{\mathbb{Z}}(M, Q).$$

Explicitly, this isomorphism sends $\eta : M \to \mathrm{Hom}_{\mathbb{Z}}(R, Q)$ to $\beta(\eta) : M \to Q$ given by $\beta(\eta)(m) = \eta(m)(1)$. Thus, $- \circ i$ becomes (note: $i : A \hookrightarrow B$ is R–linear implies that it is $\mathbb{Z}$–linear).

$$- \circ i : \mathrm{Hom}_{\mathbb{Z}}(B, Q) \to \mathrm{Hom}_{\mathbb{Z}}(A, Q),$$

which is known to be surjective since $Q \in \mathbf{Ab}$ is injective.

I_Q is a cogenerator. Now let $M \in R$-mod and $0 \neq m \in M$ be a non–zero element. We are looking for an R–linear homomorphism $f : M \to I_Q$ such that $f(m) \neq 0$. Again we use the isomorphism β given above. Let $g : M \to Q$ be a group homomorphism such that $g(m) \neq 0$ (exists since Q is a cogenerator). Let $f : M \to I_Q$ be such that $\beta(f) = g$. Then,

$$f(m) : R \to Q \text{ is such that } f(m)(1) = g(m) \neq 0.$$

Hence $f(m)$ is a non–zero element of $I_Q = \mathrm{Hom}_{\mathbb{Z}}(R, Q)$. $\square$

(18.4) Enough injectives.– Now we can prove the main theorem of today's lecture.

Theorem. *There are enough injective in* R-mod.

Remark. Recall that this meant that given any $M \in R$-mod, there exists an injective module $I_M \in R$-mod together with an injective R–linear map $\psi : M \hookrightarrow I_M$.

Proof. Let us choose an injective cogenerator I in R-mod. These exist by Proposition 18.3 above. Recall by Problem 7 of Homework 6, direct product of injective modules is injective. Consider the following direct product:

$$I_M := \prod_{\varphi \in \mathrm{Hom}_R(M,I)} I^{(\varphi)}, \qquad \text{where, } I^{(\varphi)} = I,\ \forall\ \varphi \in \mathrm{Hom}_R(R, I).$$

This module is injective, being direct product of injectives. Note that there every $I^{(\varphi)}$ comes with a natural R–linear map $\varphi : M \to I = I^{(\varphi)}$. Thus we get

$$\psi : M \to I_M, \qquad \text{given by} \qquad \psi(m) = (\varphi(m))_\varphi \in I_M.$$

We need to show that this map is injective. Here we need to use the fact that I is a cogenerator. Meaning, given non–zero element $m \in M$, there exists some $\varphi : M \to I$ such that $\varphi(m) \neq 0$. Thus φ^{th} component of $\psi(m)$ is non–zero, proving that $\psi(m) \neq 0$. $\square$