LECTURE 19

(19.0) Definition of Ext **and** Tor.– Let R be a unital commutative ring, and R-mod be the category of R–modules. Given $M, N \in R$-mod, recall that we have the following functors (see Theorems 10.4 and 12.5).

- $\mathsf{h}^N = \mathrm{Hom}_R(-, N) : R\text{-mod} \to R\text{-mod}$. It is a left exact, contravariant functor.
- $\mathsf{h}_M = \mathrm{Hom}_R(M, -) : R\text{-mod} \to R\text{-mod}$. It is a left exact, covariant functor.
- $T_N = - \otimes_R N : R\text{-mod} \to R\text{-mod}$. It is a right exact, covariant functor. Note that since $M \otimes_R N \cong N \otimes_R M$, $- \otimes_R N$ is naturally isomorphic to $N \otimes_R -$.

Recall that given a functor between two abelian categories $F : \mathcal{A} \to \mathcal{B}$, its derived functors are constructed as follows.

- *Left exact, covariant case.* Given $A \in \mathcal{A}$, choose an injective resolution $I_A^\bullet \in \mathbb{K}^\bullet(\mathcal{A})$. Then:
$$R^k F(A) := k^{\text{th}} \text{ cohomology of the cochain complex } F(I_A^\bullet).$$
- *Left exact, contravariant case.* Given $A \in \mathcal{A}$, choose a projective resolution $P_\bullet^A \in \mathbb{K}_\bullet(\mathcal{A})$. Then:
$$R^k F(A) := k^{\text{th}} \text{ cohomology of the cochain complex } F(P_\bullet^A).$$
- *Right exact, covariant case.* Given $A \in \mathcal{A}$, choose a projective resolution $P_\bullet^A \in \mathbb{K}_\bullet(\mathcal{A})$. Then:
$$L_k F(A) := k^{\text{th}} \text{ homology of the chain complex } F(P_\bullet^A).$$

  (similarly for right exact, contravariant case - but we are not going to consider it, so it is omitted here)

Remarks. (1) According to Theorem 17.3 and its corollary, injective and projective resolutions are unique up to homotopy. This, combined with the fact that H^k is the same for two homotopic morphisms (Proposition 14.3) implies that derived functors do not depend on the choice of a resolution.

(2) Derived functors are additive. To see this, let us assume F is left exact and covariant (to fix ideas). We can realize $R^k F$ as a composition of additive functors, if we replace $\mathbb{K}^\bullet(\mathcal{A})$ by its *homotopy version*, denoted here by $\mathbb{K}^\bullet(\mathcal{A})_h$.

The objects of $\mathbb{K}^\bullet(\mathcal{A})_h$ are same as those of $\mathbb{K}^\bullet(\mathcal{A})$ (namely cochain complexes), but the morphisms are changed to:
$$\mathrm{Hom}_{\mathbb{K}^\bullet(\mathcal{A})_h}(C^\bullet, D^\bullet) = \mathrm{Hom}_{\mathbb{K}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet) / \mathrm{Hom}^0_{\mathbb{K}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet),$$

where $\mathrm{Hom}^0_{\mathbb{K}^\bullet(\mathcal{A})}(C^\bullet, D^\bullet)$ is the subgroup of null–homotopic morphisms (see §14.3). By Proposition 14.3, we get that $\mathbb{K}^\bullet(\mathcal{A})_h$ is an additive category, and by Theorem 17.3, $M \mapsto I^\bullet_M$ (choice of an injective resolution) is an additive functor $\mathbb{I} : \mathcal{A} \to \mathbb{K}^\bullet(\mathcal{A})_h$. Thus, $R^kF : \mathcal{A} \to \mathcal{B}$ is the following composition of additive functors, hence additive:

$$\mathcal{A} \xrightarrow{\mathbb{I}} \mathbb{K}^\bullet(\mathcal{A})_h \xrightarrow{F} \mathbb{K}^\bullet(\mathcal{B})_h \xrightarrow{H^k} \mathcal{B}.$$

Check: an additive functor $F : \mathcal{A} \to \mathcal{B}$ applied term-by-term to cochain complexes $F : \mathbb{K}^\bullet(\mathcal{A}) \to \mathbb{K}^\bullet(\mathcal{B})$ sends null–homotopic morphisms to null–homotopic morphisms.

Thus we obtain sequences of additive functors $\{R^k\mathsf{h}^N\}, \{R^k\mathsf{h}_M\}$ and $\{L_kT_N\}$, from R-mod to itself (here $k \in \mathbb{Z}_{\geq 0}$), whose definitions and properties are the main theme of this lecture.

(19.1) Ext **functors.–** The abstract construction of derived functors, for the case of h^N and h_M can be rewritten as follows.

Definition. Given $M, N \in R$-mod, $R^k\mathsf{h}^N(M)$ is defined as follows.

- Choose a projective resolution of M:

$$\cdots \xrightarrow{d_2} P_2 \xrightarrow{d_1} P_1 \xrightarrow{d_0} P_0 \to 0.$$

- Apply $\mathrm{Hom}_R(-, N)$ to get a complex:

$$0 \to \mathrm{Hom}_R(P_0, N) \xrightarrow{d^0} \mathrm{Hom}_R(P_1, N) \xrightarrow{d^1} \mathrm{Hom}_R(P_2, N) \xrightarrow{d^2} \cdots$$

  where $d^j = - \circ d_j$.
- Take cohomology: $\boxed{R^k\mathsf{h}^N(M) = \mathrm{Ker}(d^k)/\,\mathrm{Im}(d^{k-1})}$

Similarly, $R^k\mathsf{h}_M(N)$ is defined as follows.

- Choose an injective resolution of N:

$$0 \to I^0 \xrightarrow{f^0} I^1 \xrightarrow{f^1} I^2 \xrightarrow{f^2} \cdots$$

- Apply $\mathrm{Hom}_R(M, -)$ to get a complex:

$$0 \to \mathrm{Hom}_R(M, I^0) \xrightarrow{d^0} \mathrm{Hom}_R(M, I^1) \xrightarrow{d^1} \mathrm{Hom}_R(M, I^2) \xrightarrow{d^2} \cdots$$

  where $d^j = f^j \circ -$.
- Take cohomology: $\boxed{R^k\mathsf{h}_M(N) = \mathrm{Ker}(d^k)/\,\mathrm{Im}(d^{k-1})}$

It will be shown next week that $\boxed{R^k\mathsf{h}^N(M) \cong R^k\mathsf{h}_M(N)}$. For now, we are going to assume this, and define:

$$\boxed{\mathrm{Ext}^k_R(M, N) := R^k\mathsf{h}^N(M) = R^k\mathsf{h}_M(N)}$$

Theorem. *Let $N \in R$-mod. Then we have additive, contravariant functors:*

$$\mathrm{Ext}^k_R(-, N) : R\text{-mod} \to R\text{-mod}, \qquad k \in \mathbb{Z}_{\geq 0}.$$

This sequence of functors has the following properties:

(1) $\mathrm{Ext}^0_R(-, N) = \mathrm{Hom}_R(-, N)$.

(2) *For a projective* $P \in R$-mod, $\mathrm{Ext}^k_R(P, N) = 0$, *for every* $k \geq 1$.

(3) *Given a short exact sequence* $0 \to M_1 \to M_2 \to M_3 \to 0$, *there exist connecting morphisms* $\delta^k : \mathrm{Ext}^k_R(M_1, N) \to \mathrm{Ext}^{k+1}_R(M_3, N)$, *such that the following sequence is exact:*

$$0 \to \mathrm{Hom}(M_3, N) \to \mathrm{Hom}(M_2, N) \to \mathrm{Hom}(M_1, N) \xrightarrow{\delta^0}$$
$$\to \mathrm{Ext}^1(M_3, N) \to \mathrm{Ext}^1(M_2, N) \to \mathrm{Ext}^1(M_1, N) \xrightarrow{\delta^1}$$
$$\to \mathrm{Ext}^2(M_3, N) \to \mathrm{Ext}^2(M_2, N) \to \mathrm{Ext}^2(M_1, N) \quad \cdots$$

(4) *Given two short exact sequences and morphisms making each square in the following diagram commute:*

$$\begin{array}{ccccccccc}
0 & \to & M_1 & \to & M_2 & \to & M_3 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & M_1' & \to & M_2' & \to & M_3' & \to & 0
\end{array}$$

we get the following commutative diagram, for each $k \geq 0$*:*

$$\begin{array}{ccc}
\mathrm{Ext}^k(M_1, N) & \longrightarrow & \mathrm{Ext}^{k+1}(M_3, N) \\
\uparrow & & \uparrow \\
\mathrm{Ext}^k(M_1', N) & \longrightarrow & \mathrm{Ext}^{k+1}(M_3', N)
\end{array}$$

Exercise. Write the statement of the analogous theorem for $\mathrm{Ext}^k(M, -)$ functors.

(19.2) Tor **functors.**– Let us again write in detail the construction of left derived functors of a right exact, covariant functor, in the case of $T_N = - \otimes_R N$.

Definition. Given $M, N \in R$-mod, the R–module $\mathrm{Tor}^R_k(M, N)$ is defined as follows.

- Choose a projective resolution of M:
$$\cdots \xrightarrow{f_2} P_2 \xrightarrow{f_1} P_1 \xrightarrow{f_0} P_0 \to 0.$$
- Tensor with N to get a (chain) complex:
$$\cdots \xrightarrow{d_2} P_2 \otimes N \xrightarrow{d_1} P_1 \otimes N \xrightarrow{d_0} P_0 \otimes N \to 0,$$
where $d_j = f_j \otimes \mathrm{Id}_N$.
- Take its k^{th} homology:
$$\boxed{\mathrm{Tor}^R_k(M, N) := \mathrm{Ker}(d_{k-1}) / \mathrm{Im}(d_k)}$$

Note that we may as well take a projective resolution of N and tensor (on the left) with M. The answer will be the same since $\otimes$ is commutative.

Theorem. *Let $N \in R$-mod. Then we have a sequence of additive, covariant functors:*

$$\mathrm{Tor}_k^R(-, N) : R\text{-mod} \to R\text{-mod}, \qquad k \in \mathbb{Z}_{\geq 0}.$$

(1) $\mathrm{Tor}_0^R(-, N) = - \otimes_R N$.

(2) *If P is projective, then $\mathrm{Tor}_k^R(P, N) = 0$, for every $k \geq 1$.*

(3) *For every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$, we have connecting morphisms $\delta_k : \mathrm{Tor}_{k+1}(M_3, N) \to \mathrm{Tor}_k(M_1, N)$, for every $k \geq 0$, such that the following sequence is exact:*

$$\cdots \to \mathrm{Tor}_2(M_1, N) \to \mathrm{Tor}_2(M_2, N) \to \mathrm{Tor}_2(M_3, N) \xrightarrow{\delta_1}$$
$$\mathrm{Tor}_1(M_1, N) \to \mathrm{Tor}_1(M_2, N) \to \mathrm{Tor}_1(M_3, N) \xrightarrow{\delta_0}$$
$$M_1 \otimes N \to M_2 \otimes N \to M_3 \otimes N \to 0$$

(4) *Given two short exact sequences and morphisms making each square in the following diagram commute:*

$$\begin{array}{ccccccccc} 0 & \to & M_1 & \to & M_2 & \to & M_3 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_1' & \to & M_2' & \to & M_3' & \to & 0 \end{array}$$

we get the following commutative diagram, for each $k \geq 0$:

$$\begin{array}{ccc} \mathrm{Tor}_{k+1}(M_3, N) & \longrightarrow & \mathrm{Tor}_k(M_1, N) \\ \downarrow & & \downarrow \\ \mathrm{Tor}_{k+1}(M_3', N) & \longrightarrow & \mathrm{Tor}_k(M_1', N) \end{array}$$

(19.3) Examples of Ext.–

I. Let us take $R = \mathbb{Z}$, $M = \mathbb{Z}/m\mathbb{Z}$ and $N = \mathbb{Z}$. We compute $\mathrm{Ext}_{\mathbb{Z}}^k(M, N)$ using the two methods:

(1) Projective resolution of M: $0 \to \mathbb{Z} \xrightarrow{\mu_m} \mathbb{Z} \to 0$, where μ_m is multiplication by m. Applying $\mathrm{Hom}(-, N)$ turns it into (using $\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$) $0 \to \mathbb{Z} \xrightarrow{\mu_m} \mathbb{Z} \to 0$, whose cohomology gives us:

$$\mathrm{Ext}^0(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) = \{0\}, \qquad \mathrm{Ext}^1(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z},$$

and $\mathrm{Ext}^k(M,N) = 0$ for $k \geq 2$.

(2) Injective resolution of N: $0 \to \mathbb{Q} \xrightarrow{\pi} \mathbb{Q}/\mathbb{Z} \to 0$, where π is the natural surjection. Apply $\mathrm{Hom}(M,-)$ and use the fact that $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z},\mathbb{Q}) = \{0\}$ to get (the terms are in degrees $-1,0,1,2$ respectively):

$$0 \to 0 \to \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z},\mathbb{Q}/\mathbb{Z}) \to 0.$$

Its cohomology gives exactly the same answer as before.

Remark. As we will see next week, the notation Ext stands for *extensions* and the relation can be seen in the previous example. Namely, for every $\alpha \in \mathbb{Z}$, we have a short exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{i} M_\alpha \xrightarrow{p} \mathbb{Z}/m\mathbb{Z} \to 0,$$

where M_α is the abelian group generated by two elements, e_1, e_2, subject to a relation: $me_2 = \alpha e_1$. The morphisms i and p are given by: $i(1) = e_1$ and $p(e_1) = 0, p(e_2) = \overline{1}$. It turns out that we can find an isomorphism $M_\alpha \xrightarrow{\sim} M_\beta$ which commutes with the maps i and p, if and only if $\alpha \equiv \beta \pmod{m}$.

In general, a short exact sequence $0 \to A \to B \to C \to 0$ is also called *an extension of C by A*. Thus, there are m (up to isomorphism) extensions of $\mathbb{Z}/m\mathbb{Z}$ by $\mathbb{Z}$, which is reflected in the fact that $\mathrm{Ext}^1(\mathbb{Z}/m\mathbb{Z},\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z}$.

II. $R = \mathbb{Z}$, $M = \mathbb{Z}/m\mathbb{Z}$, $N = \mathbb{Z}/n\mathbb{Z}$. Taking the projective resolution of M as above, and applying $\mathrm{Hom}(-,N)$, we get the following complex:

$$0 \to \mathbb{Z}/n\mathbb{Z} \xrightarrow{\mu_m} \mathbb{Z}/n\mathbb{Z} \to 0.$$

Now we can write down the cohomology of this complex as:

$$\mathrm{Ext}^0(M,N) = \{x \in \mathbb{Z}/n\mathbb{Z} : mx \text{ is divisible by } n\}$$

Note that this is same as $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z},\mathbb{Z}/n\mathbb{Z})$ as expected. Check: there are $\gcd(m,n)$ many elements in this hom set.

$$\mathrm{Ext}^1(M,N) = \mathbb{Z}/(m,n) = \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

(19.4) Example of Tor.– The notation Tor stands for *torsion* and is explained by the following example. Let R be an integral domain, and let $a \in R$ be a non–zero element. Let $M = R/(a)$ and $N \in R$-mod arbitrary. We begin by writing down a projective resolution of M: $0 \to R \xrightarrow{\mu_a} R \to 0$. Tensoring with N, and remembering $R \otimes_R N \cong N$, we get

$$0 \to N \xrightarrow{\mu_a} N \to 0, \qquad \text{in degrees } 2,1,0,-1.$$

Thus, we get:

$$\mathrm{Tor}_0(M,N) = N/aN = (R/(a)) \otimes_R N, \qquad \mathrm{Tor}_1(M,N) = \{x \in N : ax = 0\}.$$

Now we can clearly see $\mathrm{Tor}_1(R/(a),N)$ consists of elements which are *a–torsion* (meaning $n \in N$ so that $an = 0$). This explains the name *torsion modules* for Tor's.