LECTURE 20

(20.0) Review.– Let R be a unital, commutative ring and R-mod the category of R–modules. Last time we introduced two functors Ext and Tor, whose value on a pair $M, N \in R$-mod is computed as follows:

$\{\mathrm{Ext}^k_R(M,N)\}_{k\geq 0}$:

(1) Choose a projective resolution of M, say $P^M_\bullet \in \mathbb{K}_\bullet\,(R\text{-mod})$. Then:

$$\mathrm{Ext}^k_R(M,N) = k^{\mathrm{th}} \text{ cohomology of the cochain complex } \mathrm{Hom}(P^M_\bullet, N).$$

(2) Choose an injective resolution of N, say $I^\bullet_N \in \mathbb{K}^\bullet\,(R\text{-mod})$. Then:

$$\mathrm{Ext}^k_R(M,N) = k^{\mathrm{th}} \text{ cohomology of the cochain complex } \mathrm{Hom}(M, I^\bullet_N).$$

The proof of the fact that the two procedures give the same answer is postponed to next week.

$\{\mathrm{Tor}^R_k(M,N)\}_{k\geq 0}$:

(1) Choose a projective resolution of M, say $P^M_\bullet \in \mathbb{K}_\bullet\,(R\text{-mod})$. Then:

$$\mathrm{Tor}^R_k(M,N) = k^{\mathrm{th}} \text{ homology of the chain complex } P^M_\bullet \otimes N.$$

(2) Choose a projective resolution of N, say $P^N_\bullet \in \mathbb{K}_\bullet\,(R\text{-mod})$. Then:

$$\mathrm{Tor}^R_k(M,N) = k^{\mathrm{th}} \text{ homology of the chain complex } M \otimes P^N_\bullet.$$

Again, we will see next week why these two methods give the same answer[1]. This will show that $\mathrm{Tor}_k(M,N) = \mathrm{Tor}_k(N,M)$.

As remarked in the last lecture, we rarely use injective resolutions for computing Ext's. A few examples of projective resolutions were given in Homework 6. In this lecture we will discuss:

- Injective modules over integral domains.
- Flat modules.

(20.1) Injective modules over integral domains.– Recall Baer's criterion for injectivity (Lemma 18.1). An R–module Q is injective if, and only if, for every ideal $\mathfrak{a} \subset R$ and an R–linear map $f : \mathfrak{a} \to Q$, there exists $q \in Q$ such that $f(a) = aq$ for every $a \in \mathfrak{a}$.

Let us assume that R is an integral domain.

[1]Thanks to William Newman and Yousef Qaddura for pointing out that this is not *obviously* true, as I erraneously said.

Definition. An R–module M is said to be *torsion free*, if for every $a \in R$, the R–linear map obtained via action of a:

$$\mu_a : M \to M, \qquad \mu_a(m) = am, \ \forall\ m \in M,$$

is injective. In other words, define:

$$M_{\text{tor}} := \{m \in M : \exists a \in R \setminus \{0\}, \text{ such that } am = 0\}.$$

M is torsion free means $M_{\text{tor}} = 0$.

M is called *divisible* if μ_a is surjective, for every $a \in R \setminus \{0\}$.

Lemma. *Recall that R is assmed to be an integral domain.*

(1) *If $Q \in R$-mod is injective, then it is divisible.*

(2) *If $Q \in R$-mod is divisible and torsion free, then it is injective.*

(3) *Assuming R is a principal ideal domain (PID), then $Q \in R$-mod is injective if, and only if it is divisible.*

Proof. (1). Assume Q is injective, and let $0 \neq a \in R$. Let $\mu_a : R \to R$ be multiplication by a. Note that this map is injective, since R is an integral domain. Applying $\text{Hom}(-, Q)$, and using the fact that $\text{Hom}(R, Q) \xrightarrow{\sim} Q$, we get a sujective map (since Q is an injective module): $Q \xrightarrow{\mu_a} Q$. Hence $\mu_a : Q \to Q$ is surjective for every $a \in R \setminus \{0\}$, proving that Q is divisible.

(2). Let $Q \in R$-mod be a divisible and torsion free module. Let $\mathfrak{a} \subset R$ be a non–zero ideal and let $f : \mathfrak{a} \to Q$ be an R–linear map. We have to exhibit an element $q \in Q$ such that $f(a) = aq$ for every $a \in \mathfrak{a}$.

Choose a non–zero element $x \in \mathfrak{a}$. Since $\mu_x : Q \to Q$ is surjective, we can find $q \in Q$ such that $\mu_x(q) = f(x)$, that is, $f(x) = xq$. Now, for any $a \in \mathfrak{a}$, we have:

$$f(ax) = af(x) = axq, \qquad \text{and} \qquad f(xa) = xf(a).$$

As $ax = xa$, we conclude that $xf(a) = x(aq)$, or $x(f(a) - aq) = 0$. Since $x \neq 0$, and Q is torsion free, we get $f(a) = aq$ as needed.

(3). Now we are assuming that R is a PID. We have already seen in (1) that if Q is injective, then it is divisible. Conversely, assume that Q is divisible. Let $\mathfrak{a} \subset R$ be a non–zero ideal. Since R is a PID, $\mathfrak{a} = (x)$ for some $x \neq 0$. Using the fact that $\mu_x : Q \to Q$ is surjective, we can find $q \in Q$ such that $\mu_x(q) = f(x)$, that is $f(x) = xq$. Now for any $a \in \mathfrak{a}$, we have $a = rx$ for some $r \in R$. This implies:

$$f(a) = f(rx) = rf(x) = rxq = aq.$$

Hence $f(a) = aq$ for every $a \in \mathfrak{a}$, which implies that Q is injective. □

Remark. If R is not a PID, there exist modules which are divisible but not injective. For example, let $\mathbb{Q}(x)$ be the field of rational functions in one variable x:

$$\mathbb{Q}(x) = \left\{ \frac{p(x)}{q(x)} : p(x), q(x) \in \mathbb{Q}[x] \text{ and } q(x) \neq 0 \right\}.$$

Consider $M = \mathbb{Q}(x)/\mathbb{Z}[x]$ as a module over $R = \mathbb{Z}[x]$. This module is divisible, but not injective (Exercise).

(20.2) Flat modules.– An R–module N is said to be *flat* if $- \otimes_R N$ is an exact functor. Recall that $- \otimes N$ is always right exact (Theorem 12.5). Thus, N is flat if and only if for every injective morphism $f : M_1 \hookrightarrow M_2$, the morphism $f \otimes \mathrm{Id}_N : M_1 \otimes N \to M_2 \otimes N$ is again injective.

Flat modules play the same role for $\otimes$ as projectives and injectives do for Hom. To be more precise, we have the following properties (easy to prove - left as an exercise):

- $P \in R$-mod is projective $\iff$ $\mathrm{Ext}^1(P, N) = 0$ for every $N \in R$-mod $\iff$ $\mathrm{Ext}^k(P, N) = 0$ for every $k \in \mathbb{Z}_{\geq 1}$, $N \in R$-mod.

- $Q \in R$-mod is injective $\iff$ $\mathrm{Ext}^1(M, Q) = 0$ for every $M \in R$-mod $\iff$ $\mathrm{Ext}^k(M, Q) = 0$ for every $k \in \mathbb{Z}_{\geq 1}$, $M \in R$-mod.

For Tor functors, we have the analogous assertion:

Lemma. *Let $F \in R$-mod. Then the following are equivalent.*

(1) *F is flat.*
(2) $\mathrm{Tor}_k(M, F) = 0$ *for every $k \in \mathbb{Z}_{\geq 1}$ and $M \in R$-mod.*
(3) $\mathrm{Tor}_1(M, F) = 0$ *for every $M \in R$-mod.*

Proof. (1)$\Rightarrow$(2). Assume F is flat. Let $M \in R$-mod and let $P_\bullet$ be a projective resolution of M. Since $- \otimes F$ is an exact functor, we get an exact sequence:

$$\cdots \longrightarrow P_2 \otimes F \longrightarrow P_1 \otimes F \longrightarrow P_0 \otimes F \longrightarrow M \otimes F \to 0.$$

This shows that $\mathrm{Tor}_0(M, F) = M \otimes F$ (as is always the case), and $\mathrm{Tor}_k(M, F) = 0$ for every $k \geq 1$.

(2)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(1). For this, we use the long exact sequence for Tor functors. Given an injective morphism $f : M_1 \hookrightarrow M_2$, we have to show that $f \otimes \mathrm{Id}_N : M_1 \otimes N \to M_2 \otimes N$ is still injective. Consider the short exact sequence

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{p} M_3 = M_2/M_1 \to 0.$$

Applying $- \otimes F$ to it, we get an exact sequence:

$$\mathrm{Tor}_1(M_3, F) \to M_1 \otimes N \xrightarrow{f \otimes \mathrm{Id}_N} M_2 \otimes N \to M_3 \otimes N \to 0.$$

Since $\mathrm{Tor}_1(M_3, F) = 0$ by our hypothesis, we get that $f \otimes \mathrm{Id}_N$ is injective. $\square$

(20.3) Some examples of flat modules.–

I. *Every free module is flat. More generally, every projective module is flat.*

The statement is clear for R viewed as an R–module, since $M \otimes_R R = M$. The following lemma proves that every free module is flat. Since every projective module is a summand of a free module (i.e, if P is projective, then there exists P' such that $P \oplus P'$ is free - see Problem 2 of Homework 6), we get flatness of a projective module using this lemma again (see also Problem 8 of Homework 6):

Lemma. *Let J be a non–empty set and let $\{N_j\}_{j\in J}$ be a set of R–modules. Set $N = \bigoplus_{j\in J} N_j$. Then N is flat if, and only if, N_j is flat for every $j \in J$.*

Proof. Let $0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$ be a short exact sequence of R–modules. Since $\otimes$ distributes over $\oplus$, tensoring with N gives:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_1 \otimes N & \longrightarrow & M_2 \otimes N & \longrightarrow & M_3 \otimes N & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \bigoplus_{j\in J} M_1 \otimes N_j & \longrightarrow & \bigoplus_{j\in J} M_2 \otimes N_j & \longrightarrow & \bigoplus_{j\in J} M_3 \otimes N_j & \longrightarrow & 0
\end{array}$$

The maps in the bottom sequence are $\oplus_{j\in J}(f \otimes \mathrm{Id}_{N_j})$ and $\oplus_{j\in J}(g \otimes \mathrm{Id}_{N_j})$. Therefore, it is exact if and only if for each j,

$$0 \to M_1 \otimes N_j \to M_2 \otimes N_j \to M_3 \otimes N_j \to 0$$

is, proving the lemma. □

II. *Rings of fractions*[2].

Assume $S \subset R$ is a multiplicatively closed set (meaning, $1 \in S$, $0 \notin S$ and $a, b \in S \Rightarrow ab \in S$). Let $S^{-1}R$ denote the ring of fractions obtained from R by formally inverting elements of S. Then, $S^{-1}R$ is a flat R–module. The proof of this fact uses two things (hopefully known to the reader, otherwise review these properties - we will give a quick proof in the next lecture): (i) $S^{-1}R \otimes_R M = S^{-1}M$ and (ii) for every short exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of R–modules, the resulting sequence:

$$0 \to S^{-1}M_1 \to S^{-1}M_2 \to S^{-1}M_3 \to 0$$

is exact.

Remark. $S^{-1}R$ is rarely free. For instance, $\mathbb{Q}$ is the total ring of fractions of $\mathbb{Z}$, hence it is flat, but we know that it is not free. More generally, $S^{-1}R$ is free if, and only if $S \subset R \setminus \{0\}$ consists of invertible elements, that is $S^{-1}R = R$.

(20.4) Analogue of Baer's criterion.–

[2]Please review rings and modules of fractions to understand this very important class of examples of flat modules. We will quickly review these in the next lecture.

Theorem. *Let $N \in R$-mod. Then N is flat if, and only if, for every ideal $\mathfrak{a} \subset R$, the natural R–linear map:*

$$\mathfrak{a} \otimes_R N \to \mathfrak{a}N, \qquad a \otimes n \mapsto an,$$

is an isomorphism.

The proof of this theorem relies on the following lemma.

Lemma. *Let $N \in R$-mod. Let $\{F_j\}_{j\in J}$ be a non–empty set of R–modules. Assume that for every $j \in J$, F_j has the property:*

(P) *For every $\iota_j : K_j \hookrightarrow F_j$, $\quad \iota_j \otimes \mathrm{Id}_N : K_j \otimes N \to F_j \otimes N$ is injective.*

Then $F = \bigoplus_{j\in J} F_j$ also satisfies this property.

Let us assume this lemma for now and finish the proof of the theorem.

Proof. ($\Rightarrow$). Consider the short exact sequence of R–modules: $\mathfrak{a} \subset R \twoheadrightarrow R/\mathfrak{a}$. Assuming N is flat, we get another short exact sequence

$$0 \to \mathfrak{a} \otimes_R N \longrightarrow R \otimes_R N \longrightarrow (R/\mathfrak{a}) \otimes_R N \to 0.$$

Using the fact that $R \otimes_R N = N$ and $(R/\mathfrak{a}) \otimes_R N = N/\mathfrak{a}N$, we get:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{a} \otimes N & \longrightarrow & R \otimes N & \longrightarrow & (R/\mathfrak{a}) \otimes N & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \| & & \\
0 & \longrightarrow & \mathfrak{a}N & \longrightarrow & N & \longrightarrow & N/\mathfrak{a}N & \longrightarrow & 0
\end{array}$$

It is easy to see that the leftmost vertical map has to be an isomorphism (both $\mathfrak{a} \otimes N$ and $\mathfrak{a}N$ are the kernel of $N \to N/\mathfrak{a}N$).

($\Leftarrow$). Let $f : M_1 \hookrightarrow M_2$ be an injective morphism of R–modules. We want to prove that $f \otimes \mathrm{Id}_N$ is also injective. By our hypothesis, R (viewed as an R–module) has property (P) from the lemma above. Using the lemma, we know that every free module has property (P).

Let $\pi : F \twoheadrightarrow M_2$ be a surjective R–linear map from a free module F. Let $F_1 := \pi^{-1}(M_1) \subset F$, $K = \mathrm{Ker}(\pi) \subset F$ and $\pi_1 = \pi|_{F_1} : F_1 \to M_1$ (again surjective). Note that $K = \pi^{-1}(\{0\}) \subset F_1$ hence $\mathrm{Ker}(\pi_1) = K$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \xrightarrow{i_1} & F_1 & \xrightarrow{\pi_1} & M_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow j & & \downarrow f & & \\
0 & \longrightarrow & K & \xrightarrow{i} & F & \xrightarrow{\pi} & M_2 & \longrightarrow & 0
\end{array}$$

Here f and j are injective maps. Tensoring this diagram with N, using the fact that tensor is right exact, and that F has property (P), we get the following diagram with exact rows:

$$\begin{array}{ccccccccc} & & K\otimes N & \longrightarrow & F_1\otimes N & \longrightarrow & M_1\otimes N & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle j\otimes \mathrm{Id}} & & \downarrow{\scriptstyle f\otimes \mathrm{Id}} & & \\ 0 & \longrightarrow & K\otimes N & \longrightarrow & F\otimes N & \longrightarrow & M_2\otimes N & \longrightarrow & 0 \end{array}$$

Note that (again by property (P) of F) the middle vertical map $j\otimes \mathrm{Id}$ is injective. Now the injectivity of $f\otimes \mathrm{Id}$ follows from the snake lemma (or an easy diagram chase). □

(20.5) Proof of Lemma (20.4).– We begin by proving this lemma for the case of two modules F_1, F_2. So $F = F_1\oplus F_2$. Let $K\subset F$ and set $K_1 := F_1\cap K\subset F_1$, and $K_2 = p_2(K)\subset F_2$ (here $p_2 : F\to F_2$ is the natural projection). We get the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1 & \longrightarrow & K & \longrightarrow & K_2 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_1 & \longrightarrow & F & \longrightarrow & F_2 & \longrightarrow & 0 \end{array}$$

Tensoring with N gives the following commutative diagram (note that the bottom exact row is split, so it remains exact after applying $-\otimes N$):

$$\begin{array}{ccccccccc} & & K_1\otimes N & \longrightarrow & K\otimes N & \longrightarrow & K_2\otimes N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_1\otimes N & \longrightarrow & F\otimes N & \longrightarrow & F_2\otimes N & \longrightarrow & 0 \end{array}$$

Since the leftmost and rightmost vertical maps are injective, by property (P) of F_1 and F_2, so must be the middle vertical map, proving (P) for F.

By an induction argument, the lemma follows when $|J| < \infty$. If J is arbitrary, we argue by contradiction as follows. Assume that there exists $\iota : K\hookrightarrow F$ such that $\iota\otimes \mathrm{Id}_N : K\otimes N\to F\otimes N$ is not injective. That means, there exists a non–zero element

$$K\otimes N \ni \sum_{i=1}^{r} k_i\otimes n_i \mapsto 0.$$

Now we can restrict our attention to a finite subset $J_0\subset J$. Let $p_\ell : F = \oplus_{j\in J}F_j\to F_\ell$ be the natural map (induced from $\delta_{\ell j} : F_j\to F_\ell$).

$$J_0 := \{\ell\in J : p_\ell(k_i)\neq 0,\ \text{for some } 1\leq i\leq r\}.$$

It is easy to see that J_0 is a finite set. For each $\ell\in J_0$, take $K_\ell\subset F_\ell$ to be the submodule generated by $\{p_\ell(k_i) : 1\leq i\leq r\}$. The relation given above contradicts the fact that the lemma holds for finite indexing sets.