LECTURE 21

(21.0) Flat modules.– Let R be a unital, commutative ring. Recall that last time we gave the following definition: $N \in R$-mod is *flat* if $- \otimes_R N$ is an exact functor. Further, we showed that N being flat is equivalent to:

(1) For every injective R–linear map $f : M_1 \hookrightarrow M_2$, $f \otimes \mathrm{Id}_N : M_1 \otimes N \to M_2 \otimes N$ is injective.

(2) For every ideal $\mathfrak{a} \subset R$, the natural map $\mathfrak{a} \otimes N \to \mathfrak{a}N$ is an isomorphism.

(3) $\mathrm{Tor}_k(M, N) = 0$, for every $k \in \mathbb{Z}_{\geq 1}$ and $M \in R$-mod.

(4) $\mathrm{Tor}_1(M, N) = 0$, for every $M \in R$-mod.

In this lecture, we will prove that *flatness is a local property* (see §21.R5 for what it means), and *every finitely presented flat module over a local ring is free* (see §21.R4 for the definition of local rings). Sections 21.R1–21.R6 are included as a review of a few key points about localizations.

(21.1) Flatness is a local property.– Again, let $N \in R$-mod.

Lemma. *The following are equivalent.*

(1) *N is a flat R–module.*
(2) *$N_{\mathfrak{p}}$ is a flat $R_{\mathfrak{p}}$–module, for every prime ideal $\mathfrak{p}$.*
(3) *$N_{\mathfrak{m}}$ is a flat $R_{\mathfrak{m}}$–module, for every maximal ideal $\mathfrak{m}$.*

The proof of this lemma is obtained easily by using Corollary 21.R5 (exactness is a local property) and the fact that the operation of inverting elements from a multiplicatively closed set distributes over $\otimes$ (Proposition 21.R2).

(21.2) Finitely presented modules.– Recall that a module $M \in R$-mod is said to be *finitely generated* if there exists a finite set of element $\{m_1, \ldots, m_p\} \subset M$ such that $M = Rm_1 + \cdots + Rm_p$ (just a sum, not a direct sum). In other words, we have a surjective R–linear map from a free module of finite rank $F = R^p$ to M.

Definition. An R–module M is said to be finitely presented, if there exist two free modules of finite rank F_1, F_0 and an R–linear map $F_1 \xrightarrow{r} F_0$ such that $M = \mathrm{CoKer}(r)$. Meaning, we have an exact sequence (called *finite presentation* of M)

$$F_1 \xrightarrow{r} F_0 \xrightarrow{\pi} M \to 0.$$

This signifies that (i) M is finitely generated, so that we have a surjective R–linear map $F_0 \to M$ from a finite rank free module, and (ii) $K = \mathrm{Ker}(\pi)$ - the module of relations among generators of M - is also finitely generated. Thus, M can be given a *presentation* by a finite set of generators and a finite set of relations.

Lemma. *Let M be a finitely presented module. Then for every short exact sequence*

$$0 \to K \xrightarrow{f} E \xrightarrow{g} M \to 0$$

such that E is finitely generated, we have that K is also finitely generated.

Remark. This lemma is true without finitely presented hypothesis for *Noetherian rings.* However, if the ring is not Noetherian, for instance, let k be a field and $R = k[x_i : i \in I]$ for an infinite set I, we can easily construct a counterexample, if finitely presented hypothesis is omitted. Take $M = k = R/(x_i : i \in I)$, $E = R$ (finitely generated). Then K is generated by an infinite set of elements $\{x_i : i \in I\}$.

Proof. Let F_0, F_1 be two free modules of finite rank which give a finite presentation of M:

$$F_1 \xrightarrow{r} F_0 \xrightarrow{\pi} M \to 0.$$

Since $g : E \to M$ is surjective, and F_0 is projective, we get an R–linear map $\psi : F_0 \to E$ such that $g \circ \psi = \pi$. The composition $g \circ \psi \circ r : F_1 \to M$ is zero, since $\pi \circ r = 0$. Thus, $\psi \circ r$ factors through the kernel of g, namely $K \xrightarrow{f} E$. Let us call it ϕ:

$$\begin{array}{ccccccccc} & & F_1 & \xrightarrow{r} & F_0 & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \Vert & & \\ 0 & \longrightarrow & K & \xrightarrow{f} & E & \xrightarrow{g} & M & \longrightarrow & 0 \end{array}$$

Therefore, we obtain an isomorphism $K/\phi(F_1) \xrightarrow{\sim} E/\psi(F_0)$, proving that $K/\phi(F_1)$ is finitely generated (because E is finitely generated, and a quotient of finitely generated module is again finitely generated). Note that $\phi(F_1)$ is again finitely generated since F_1 is. Thus the two ends of the following short exact sequence are finitely generated:

$$0 \to \phi(F_1) \longrightarrow K \longrightarrow K/\phi(F_1) \to 0,$$

which proves that so must be K. □

(21.3) Flat modules over local rings.– Now assume that $(A, \mathfrak{m})$ is a local ring. Let M be an A–module.

Theorem. *If M is a finitely presented module, such that the natural R–linear map $\mathfrak{m} \otimes M \to M$ is injective, then M is free.*

More precisely, view $M/\mathfrak{m}M$ as a finite–dimensional vector space over the field $k = A/\mathfrak{m}$. Given any basis $\{x_1, \ldots, x_p\}$ of $M/\mathfrak{m}M$, and any lifts:

$$m_i \in M \text{ such that under } \pi : M \to M/\mathfrak{m}M, \text{ we have } \pi(m_i) = x_i, \ \forall\ 1 \le i \le p,$$

M is freely generated as an A–module by $\{m_1, \ldots, m_p\}$.

Remark. Note that the hypothesis $\mathfrak{m} \otimes M \to M$ is injective holds, by definition, for flat modules (see Theorem (20.4)). Thus, we have the statement *finitely presented flat modules are locally free*. It turns out that finitely presented, locally free modules are exactly projective modules. We are not going to prove this, but the reader can consult any textbook on Commutative algebra for this, for instance *D. Eisenbud, Commutative algebra with a view toward algebraic geometry, Exercise 4.11, page 136.*

Proof. We begin by showing that M is generated by $\{m_1, \ldots, m_p\}$. Let $M' \subset M$ be the submodule generated by this set, and let $K = M/M'$ (finitely generated, being a quotient of one). By our hypothesis $\mathfrak{m}K = K$, which by Nakayama's lemma (see Lemma 20.R6 below) means that $K = 0$.

Now we show that there are no relations among $\{m_i\}_{i=1}^{p}$. Define the R–linear map $\pi : F = R^p \to M$ given by $e_i \mapsto m_i$. Since M is finitely presented, Lemma (21.2) implies that the kernel $L = \mathrm{Ker}(\pi)$ is again finitely generated. Tensoring the short exact sequence $0 \to L \xrightarrow{i} F \xrightarrow{\pi} M \to 0$ with $\mathfrak{m}$, and using the fact that tensor is right exact, we get:

$$
\begin{array}{ccccccccc}
 & & \mathfrak{m} \otimes L & \longrightarrow & \mathfrak{m} \otimes F & \longrightarrow & \mathfrak{m} \otimes M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

where the vertical maps are coming from the natural action $a \otimes x \mapsto ax$, for every $a \in \mathfrak{m}$ and x in the module under consideration.

Snake lemma applies, and we get an exact sequence, using the hypothesis that $\mathrm{Ker}(\mathfrak{m} \otimes M \to M) = 0$.

$$0 \to L/\mathfrak{m}L \longrightarrow F/\mathfrak{m}F \longrightarrow M/\mathfrak{m}M.$$

Note that both $F/\mathfrak{m}F$ and $M/\mathfrak{m}M$ are p–dimensional vector spaces over $k = A/\mathfrak{m}$, with basis $\overline{e_i} \mapsto x_i$. Hence the last morphism is an isomorphism, proving that $L/\mathfrak{m}L = 0$. Nakayama's lemma applies since L is finitely generated, and we get $L = 0$. In conclusion, $F \xrightarrow{\sim} M$ and therefore M is free. □

(21.R1) Rings and modules of fractions.– Let R be a unital, commutative ring and let $S \subset R$. Recall that S is said to be *multiplicatively closed* if (i) $0 \notin S$, (ii) $1 \in S$, and (iii) $a, b \in S \Rightarrow ab \in S$.

Ring of fractions. $S^{-1}R$ is defined, as a set: $S^{-1}R := S \times R/\sim$, where $(s_1, r_1) \sim (s_2, r_2)$ if there exists $t \in S$ such that $t(s_2r_1 - s_1r_2) = 0$. It is a routine exercise to show that $\sim$ is an equivalence relation, and the following operations give a structure of a unital commutative ring on the set $S^{-1}R$. As usual, we write a typical element of $S^{-1}R$ as a fraction $\frac{r}{s}$.

- $\dfrac{r_1}{s_1} + \dfrac{r_2}{s_2} = \dfrac{s_2r_1 + s_1r_2}{s_1s_2}$.
- $\dfrac{r_1}{s_1} \cdot \dfrac{r_2}{s_2} = \dfrac{r_1r_2}{s_1s_2}$.

Let $j_S : R \to S^{-1}R$ denote the ring homomorphism $r \mapsto \frac{r}{1}$.

Module of fractions. Analogously, given an R–module M, we define $S^{-1}M$, as a set: $S \times M/\sim$, where $(s_1, m_1) \sim (s_2, m_2)$ if there exists $t \in S$ such that $t(s_1m_2 - s_2m_1) = 0$. Again, the fraction $\dfrac{m}{s}$ represents the equivalence class of the pair (s, m). The following operations give a structure of an $S^{-1}R$–module on the set $S^{-1}M$:

- (Abelian group structure). $\dfrac{m_1}{s_1} + \dfrac{m_2}{s_2} = \dfrac{s_2m_1 + s_1m_2}{s_1s_2}$.
- (Action of $S^{-1}R$). $\dfrac{r}{s} \cdot \dfrac{m}{t} = \dfrac{rm}{st}$.

For an R–linear map $f : M \to N$, we get an $S^{-1}R$–linear map, again denoted by f,

$$f : S^{-1}M \to S^{-1}N, \qquad f\left(\frac{m}{s}\right) = \frac{f(m)}{s}.$$

In conclusion, we have a functor $S^{-1}(-) : R\text{-mod} \to S^{-1}R\text{-mod}$. Morever, the ring homomorphim $j_S : R \to S^{-1}R$ allows us to view an $S^{-1}R$–module as an R–module.

(21.R2) Basic properties of $S^{-1}(-)$.– The following list of properties follow easily from definitions.

Proposition. (1) *For every $M \in R$-mod, $S^{-1}R \otimes_R M = S^{-1}M$.*

(2) *$S^{-1}(-) : R\text{-mod} \to S^{-1}R\text{-mod}$ is an exact functor. Namely, for every exact sequence $0 \to M' \to M \to M'' \to 0$, the resulting sequence of $S^{-1}R$–modules (or R–modules via j_S):*

$$0 \to S^{-1}M' \to S^{-1}M \to S^{-1}M'' \to 0,$$

is exact.

(3) *For every $M, N \in R$-mod, we have:*

$$(S^{-1}M) \otimes_{S^{-1}R} (S^{-1}N) = S^{-1}(M \otimes_R N) = (S^{-1}M) \otimes_R N.$$

Properties (1) and (2) imply that $S^{-1}R$, viewed as an R–module, is flat.

(21.R3) Localization.– Recall the definitions of *prime* and *maximal* ideals.

Definition. An ideal $\mathfrak{p} \subsetneq R$ is said to be prime, if $ab \in \mathfrak{p}$ implies $a \in \mathfrak{p}$, or $b \in \mathfrak{p}$. Alternately, $R/\mathfrak{p}$ is an integral domain. The contrapositive of the definition gives that $\mathfrak{p} \subsetneq R$ is a prime ideal if, and only if $R \setminus \mathfrak{p}$ is multiplicatively closed.

An ideal $\mathfrak{m} \subsetneq R$ is said to be maximal, if it is maximal with respect to inclusion. That is, for any ideal $\mathfrak{a}$, $\mathfrak{m} \subset \mathfrak{a} \subset R$ means either $\mathfrak{a} = \mathfrak{m}$, or $\mathfrak{a} = R$. In other words, $R/\mathfrak{m}$ has no non–zero ideals, and hence is a field. This is the quickest way to see that every maximal ideal is prime.

It is a Zorn's lemma style proof that shows that given any proper ideal $\mathfrak{a} \subsetneq R$, there is some maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$. In particular, maximal ideals (and hence prime ideals) exist.

Note: I am always assuming that prime and maximal ideals are proper.

Given a prime ideal $\mathfrak{p}$ of R, the ring of fractions obtained by taking $S = R \setminus \mathfrak{p}$, is denoted by $R_\mathfrak{p}$, and is called *localization of* R *at* $\mathfrak{p}$. More explicitly, elements of $R_\mathfrak{p}$ are of the form $\frac{r}{s}$ where $r \in R$ and $s \notin \mathfrak{p}$. The terminology comes from the notion of local rings, as we recall below.

(21.R4) Local rings.– A ring A is said to be *local* if it has only one maximal ideal $\mathfrak{m} \subsetneq A$. Often we say that $(A, \mathfrak{m})$ is a local ring, to indicate the unique maximal ideal. One way to show that $\mathfrak{m} \subsetneq A$ is the only maximal ideal in A, is to prove that $A \setminus \mathfrak{m} = A^\times$, the group of invertible elements of A.

Lemma. *Let R be a unital commutative ring, and let $\mathfrak{p} \subsetneq R$ be a prime ideal. Then the ring of fractions $A = R_\mathfrak{p}$ is a local ring, with unique maximal ideal $\mathfrak{p}R_\mathfrak{p}$.*

Proof. The proof relies on the following bijection, which describes ideals in a ring of fractions $S^{-1}R$:

$$\left\{\text{Proper ideals in } S^{-1}R\right\} \leftrightarrow \{\text{Ideal } \mathfrak{a} \subsetneq R \text{ such that } S \cap \mathfrak{a} = \emptyset\}.$$

This bijection takes an ideal $\mathfrak{a} \subset R$ and sends it to $S^{-1}\mathfrak{a} = \{\frac{a}{s} : a \in \mathfrak{a}, s \in S\} \subset S^{-1}R$.

Now we specialize to our case, when $S = R \setminus \mathfrak{p}$. Thus,

$$A = S^{-1}R, \qquad \mathfrak{m} = S^{-1}\mathfrak{p} = \{\frac{p}{s} : p \in \mathfrak{p}, s \in S\} \subsetneq A.$$

We claim that $\mathfrak{m}$ is a maximal ideal of A. If $x \in A \setminus \mathfrak{m}$, then $x = \frac{r}{s}$, where $r, s \notin \mathfrak{p}$. But then $\frac{s}{r}$ is also in A and is inverse of x. This proves that every element outside of $\mathfrak{m}$ is a unit, hence $\mathfrak{m}$ is the maximal ideal. □

(20.R5) Exactness is a local property.– In commutative algebra, a property of modules over R is called a *local property* if its verification on $M \in R$-mod is equivalent to that on $M_\mathfrak{p} \in R_\mathfrak{p}$-mod for every prime ideal (resp. for every maximal ideal). For instance, the following lemma says that being 0 is a local property.

Lemma. *Let $M \in R$-mod. Then the following conditions are equivalent.*

(1) $M = 0$.
(2) $M_\mathfrak{p} = 0$ *for every prime ideal* $\mathfrak{p} \subsetneq R$.
(3) $M_\mathfrak{m} = 0$ *for every maximal ideal* $\mathfrak{m} \subsetneq R$.

Proof. (1)⇒(2)⇒(3) are obvious implications. To show (3)⇒(2), assume that $M_\mathfrak{m} = 0$ for every maximal ideal $\mathfrak{m} \subsetneq R$. For the sake of a contradiction, let us assume that $M \neq 0$, and let $m \in M$. Then

$$\mathrm{Ann}(m) := \{r \in R : rm = 0\} \subset R,$$

is a proper ideal, since $1 \notin \mathrm{Ann}(m)$. So it must be contained in some maximal ideal, say $\mathfrak{m} \supset \mathrm{Ann}(m)$. We claim that $\frac{m}{1} \in M_{\mathfrak{m}}$ is a non–zero element, contradicting the fact that $M_{\mathfrak{m}} = 0$. To see this, if $\frac{m}{1} = 0$, there must exist some $s \in R \setminus \mathfrak{m}$ such that $sm = 0$ (by definition of the equivalence relation). But that means $s \in \mathrm{Ann}(m) \subset \mathfrak{m}$, and $s \notin \mathfrak{m}$, which is a contradiction. □

This same argument shows that the property of a short sequence to be exact is also local. That is,

Corollary. *Let M', M, M'' be three R–modules. Then the following are equivalent for a pair of morphisms $M' \xrightarrow{f} M$ and $M \xrightarrow{g} M''$.*

(1) $0 \to M' \to M \to M'' \to 0$ *is exact.*
(2) $0 \to M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}} \to 0$ *is exact, for every prime ideal* $\mathfrak{p}$.
(3) $0 \to M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \to M''_{\mathfrak{m}} \to 0$ *is exact, for every maximal ideal* $\mathfrak{m}$.

(21.R6) Nakayama's lemma.– Assume that $(A, \mathfrak{m})$ is a local ring and $K \in A$-mod is a finitely–generated A–module. The following result is absolutely fundamental in the theory of local rings.

Lemma. *If $K = \mathfrak{m}K$ then $K = 0$.*

Proof. Let $\{x_1, \ldots, x_p\} \subset K$ be a finite set of generators of K. By our hypothesis, there exists elements $a_{ij} \in \mathfrak{m}$, $1 \leq i, j \leq p$, such that

$$x_i = \sum_{j=1}^{p} a_{ij} x_j, \qquad \forall\ 1 \leq i \leq p.$$

Thus, the element $D = \mathrm{Det}(\mathrm{Id} - (a_{ij})) \in A$ annihilates all $x_i's$ (Cayley–Hamilton theorem), proving that $D \cdot x = 0$ for every $x \in K$. But $D \notin \mathfrak{m}$, so it must be a unit. This implies $x = 0$ for every $x \in K$, as claimed. □