LECTURE 22

(22.0) Review.– Let R be a unital, commutative ring. We defined *flat modules* as: $N \in R$-mod is said to be flat if $-\otimes N$ is an exact functor. Last time we prove the following two results:

- The following three assertions are equivalence.
  - N is flat.
  - $N_{\mathfrak{p}}$ is flat for every prime ideal $\mathfrak{p} \subsetneq R$.
  - $N_{\mathfrak{m}}$ is flat for every maximal ideal $\mathfrak{m} \subsetneq R$.
- If $(A, \mathfrak{m})$ is a local ring, and N is a finitely presented flat A–module, then N is free.

Remark. Geometrically, flat morphisms correspond to *locally trivial fibrations* where dimensions (or number of elements) of fibers does not change. For instance, if X and Y are connected, complex manifolds, and $f : X \to Y$ is an analytic map, then f being flat implies $f(X) \subset Y$ is open, and for every $y \in f(X)$, $X_y = f^{-1}(y)$ has the same dimension (independent of y).

(22.1) Ext**,** Tor **and loose ends.–** Recall that we started (since Lecture 13) with an abstract notion of derived functors. Let $F : \mathcal{A} \to \mathcal{B}$ is an additive functor between two abelian categories.

- If F is left exact (covariant or contravariant), we get a sequence of functors $\{R^kF : \mathcal{A} \to \mathcal{B}\}_{k\geq 0}$ (same variance as F) called *right derived functors of* F.
- If F is right exact, we get $\{L_kF : \mathcal{A} \to \mathcal{B}\}_{k\geq 0}$ called *left derived functors of* F.

When $\mathcal{A} = \mathcal{B} = R$-mod, and $M, N \in R$-mod, we get the following derived functors.

$\mathrm{Ext}^k(M, N)$ can be viewed as $R^k\mathsf{h}_M(N)$ or $R^k\mathsf{h}^N(M)$. Here, $\mathsf{h}_M = \mathrm{Hom}(M, -)$ and $\mathsf{h}^N = \mathrm{Hom}(-, N)$ are covariant and contravariant Hom functors. They are both left exact.

$\mathrm{Tor}_k(M, N)$ can be viewed as $L_kT_M(N)$ or $L_kT^N(M)$. Here, $T_M = M\otimes -$ and $T^N = -\otimes N$ are both covariant, right exact functors (since $\otimes$ is commutative, $T^N = T_N$).

We still have to prove that the two ways of defining Ext and Tor give the same result. This is the main theorem of these notes.

(22.2) The case of Ext**.–** For definiteness, let us start with $R^k\mathsf{h}_M(N)$ and show that it is the same as $R^k\mathsf{h}^N(M)$. In more detail, we will take the following definition as the starting point, and call the resulting functor $E^k(M, N)$.

Given $N \in R$-mod, choose an injective resolution of N:

$$0 \to I^0 \xrightarrow{i^0} I^1 \xrightarrow{i^1} I^2 \xrightarrow{i^2} \cdots \qquad \text{exact at } I^k,\ \forall k \in \mathbb{Z}_{\geq 1}, \text{ and } \operatorname{Ker}(i^0) = N.$$

Define $E^k(M,N)$ as k^{th} cohomology of the complex $\operatorname{Hom}(M, I^\bullet)$. It is a covariant functor in the variable N.

$$\boxed{E^k(M,N) = H^k(\operatorname{Hom}(M, I^\bullet))}$$

Recall the following properties of $E^k(M,N)$, which will be needed for our proof.

(1) $E^0(M,N) = \operatorname{Hom}(M,N)$. This follows from the left exactness of $\operatorname{Hom}(M,-)$.

(2) If N is injective, then $E^k(M,N) = 0$ for every $k \geq 1$, $M \in R$-mod. This is true because we can take $0 \to N \to 0$ as an injective resolution of N.

(3) For every short exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$, we get a connecting homomorphism $E^k(M,N_3) \to E^{k+1}(M,N_1)$, making the following sequence exact:

$$\cdots \to E^k(M,N_1) \to E^k(M,N_2) \to E^k(M,N_3) \to E^{k+1}(M,N_1) \to E^{k+1}(M,N_2) \to \cdots$$

Lemma. *Assume that $0 \to N \to I \to N' \to 0$ is a short exact sequence of R–modules, where I is injective. Then we have:*

$$E^{k+1}(M,N) = E^k(M,N'), \quad \text{for every } k \geq 1,$$

and $E^1(M,N) = \operatorname{CoKer}(\operatorname{Hom}(M,I) \to \operatorname{Hom}(M,N'))$.

Proof. The proof follows easily by the long exact sequence in cohomology. Namely,

$$0 \to \operatorname{Hom}(M,N) \to \operatorname{Hom}(M,I) \to \operatorname{Hom}(M,N') \to E^1(M,N) \to E^1(M,I) = 0.$$

The last term is zero since I is injective. Thus we get:

$$E^1(M,N) = \operatorname{CoKer}(\operatorname{Hom}(M,I) \to \operatorname{Hom}(M,N')).$$

Now for $k \geq 1$, the following is a part of the long exact sequence:

$$0 = E^k(M,I) \to E^k(M,N') \to E^{k+1}(M,N) \to E^{k+1}(M,I) = 0,$$

which proves that $E^k(M,N') \cong E^{k+1}(M,N)$. □

(22.3) $R^k\mathsf{h}_M(N) = R^k\mathsf{h}^N(M)$.– Now we are ready to state and prove the desired result. Let $P_\bullet$ be a projective resolution of M. That is, we have a chain complex:

$$\cdots \xrightarrow{p_2} P_2 \xrightarrow{p_1} P_1 \xrightarrow{p_0} P_0 \to 0,$$

which is exact at P_k for every $k \geq 1$, and $\operatorname{CoKer}(p_0) = M$. In addition to the properties of $E^k(M,N)$ listed above, we are going to use the fact that $\operatorname{Hom}(P,-)$ is exact for P a projective R–module.

Theorem. $E^k(M,N) = H^k(\operatorname{Hom}(P_\bullet, N))$.

Proof. The proof is an induction argument on k. For $k = 0$, we have

$$E^0(M,N) = \mathrm{Hom}(M,N) = H^0(\mathrm{Hom}(P_\bullet, N)), \text{ for every } N \in R\text{-mod},$$

again by left exactness of Hom functors.

Assume that we know $E^k(M,X) = H^k(\mathrm{Hom}(P_\bullet, X))$ for every $X \in R$-mod and $0 \leq k \leq \ell$, where $\ell \in \mathbb{Z}_{\geq 0}$.

Consider a short exact sequence of R–modules, where I is injective:

$$0 \to N \xrightarrow{f} I \xrightarrow{g} N' \to 0.$$

Applying $\mathrm{Hom}(P_\bullet, -)$ to this short exact sequence, and using the fact that each P_j is projective, so $\mathrm{Hom}(P_j, -)$ is exact, we get get a short exact sequence of cochain complexes:

$$0 \to \mathrm{Hom}(P_\bullet, N) \to \mathrm{Hom}(P_\bullet, I) \to \mathrm{Hom}(P_\bullet, N') \to 0.$$

This gives rise to a long exact sequence in cohomology (see Lecture 15, Theorem 15.1). Now we are ready to carry out the induction step.

Case 1: $\ell = 1$. Using $H^0(\mathrm{Hom}(P_\bullet), X) = \mathrm{Hom}(M, X)$, the first part of the long exact sequence in cohomology takes the following form:

$$0 \to \mathrm{Hom}(M,N) \to \mathrm{Hom}(M,I) \to \mathrm{Hom}(M,N') \to H^1(\mathrm{Hom}(P_\bullet, N)) \to H^1(\mathrm{Hom}(P_\bullet, I)).$$

The last term is zero, since I is injective, so $\mathrm{Hom}(-, I)$ is exact. This implies that:

$$H^1(\mathrm{Hom}(P_\bullet, N)) = \mathrm{CoKer}(\mathrm{Hom}(M,I) \to \mathrm{Hom}(M,N')) = E^1(M,N),$$

by Lemma 22.2 above.

Case 2: $\ell > 1$. Again we focus on a part of the long exact sequence, and use the fact that I being injective, we have $H^k(\mathrm{Hom}(P_\bullet, I)) = 0$ for every $k \geq 1$.

$$0 = H^\ell(P_\bullet, I) \to H^\ell(P_\bullet, N') \to H^{\ell+1}(P_\bullet, N) \to H^{\ell+1}(P_\bullet, I) = 0.$$

Thus, we obtain:

$$H^{\ell+1}(P_\bullet, N) = H^\ell(P_\bullet, N') = E^\ell(M, N') = E^{\ell+1}(M,N),$$

where in the second equality we have used the induction hypothesis, and in the third, Lemma 22.2. □

(22.4) The case of Tor.**–** Exact same argument as in the proof of the theorem above works (details are left as an instructive exercise to the reader) to prove that:

$$\boxed{\mathrm{Tor}_k(M,N) = H_k(P_\bullet^M \otimes N) = H_k(M \otimes P_\bullet^N)}$$

Here, $P_\bullet^M$ and $P_\bullet^N$ are projective resolutions of M and N respectively.

In fact, since in the proof of the theorem, we only used the fact that $\mathrm{Hom}(P,-)$ is exact, it allows applies to prove that

$$\boxed{\mathrm{Tor}_k(M,N) = H_k(F_\bullet^M \otimes N)}$$

where $F_\bullet^M$ is a *flat resolution* of M. Compare this argument with Exercise 11 of Homework 7.

Example. Assume that R is an integral domain, and K is its field of fractions (that is, $K = (R \setminus \{0\})^{-1}R$). The following is a flat resolution of $M = K/R$:

$$0 \to R \xrightarrow{j} K \to 0,$$

where $j : R \to K$ is the canonical inclusion. Using this resolution, we can compute $\mathrm{Tor}_k(K/R, N)$ for $N \in R$-mod as follows. Tensoring the above complex with N (and using $R \otimes_R N = N$) gives:

$$0 \to N \xrightarrow{j_N} K \otimes_R N \to 0, \qquad \text{degrees } 2, 1, 0, -1 \text{ respectively.}$$

Thus $\mathrm{Tor}_0(K/R, N) = N_K/j_N(N)$ (where $N_K = K \otimes_R N$ is a K–vector space and $j_N(N)$ is its R–submodule).

$$\mathrm{Tor}_1(K/R, N) = \mathrm{Ker}(j_N) = \{n \in N : \exists\, 0 \neq a \in R \text{ such that } an = 0\} = N_{\mathrm{tor}}.$$

For instance, take $R = \mathbb{Z}$, so that $K = \mathbb{Q}$. A typical finitely–generated abelian group has the following form:

$$N = \mathbb{Z}^{\oplus r} \oplus (\mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_s\mathbb{Z}).$$

Then, the computation above gives:

$$\mathrm{Tor}_0(\mathbb{Q}/\mathbb{Z}, N) = (\mathbb{Q}/\mathbb{Z})^{\oplus r}, \qquad \mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, N) = \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_s\mathbb{Z}.$$