LECTURE 23

(23.0) General remarks and outline.– This is the last lecture of the segment *homological algebra* of our course. We will end with a brief discussion (without proofs) of *spectral sequences.*

Last time we talked about why the two ways of computing Ext (resp. Tor) give the same answer. The fundamental theorem of spectral sequences (associated to a bicomplex) can be used to get a different proof. These notes only contain minimal definitions required to state this theorem. This material is not part of the course. The reader can consult `https://math.stanford.edu/~vakil/0708-216/216ss.pdf` for a friendly introduction. *R. Bott and W. Tu : Differential forms in algebraic topology* contains a very thorough treatment of spectral sequences.

(23.1) Bicomplex and its total cohomology.– Let $\mathcal{A}$ be an abelian category. A *bicomplex valued in* $\mathcal{A}$ is the data of:

- A set of objects of $\mathcal{A}$: $\{K^{p,q}\}_{p,q\in\mathbb{Z}_{\geq 0}}$.
- Horizontal differentials $d : K^{p,q} \to K^{p+1,q}$ (i.e, $d \circ d = 0$).
- Vertical differentials $\partial : K^{p,q} \to K^{p,q+1}$.

This data is subject to the condition that $d\partial = \partial d$, that is, the following diagram commutes:

$$\begin{array}{ccc} K^{p,q+1} & \xrightarrow{d} & K^{p+1,q+1} \\ \uparrow{\scriptstyle\partial} & & \uparrow{\scriptstyle\partial} \\ K^{p,q} & \xrightarrow{d} & K^{p+1,q} \end{array}$$

The *total complex* of a bicomplex $\{K^{p,q}\}$ is defined as:

$$K^{[n]} = \bigoplus_{p+q=n} K^{p,q}, \qquad D = d + (-1)^p\partial \text{ on } K^{p,q}.$$

$D \circ D = 0$. This is because, on $K^{p,q}$, we have:

$$D \circ D = d \circ d + \partial \circ \partial + (-1)^{p+1}\partial \circ d + d \circ ((-1)^p\partial) = 0.$$

Let us denote by $H^n(K^{[\bullet]})$ the n^{th}–cohomology group of the total complex. We will write $H^n_D(K^{[\bullet]})$ is dependence on the differential needs to be specified. These are called *total cohomology groups of a bicomplex* $K^{\bullet,\bullet}$.

(23.2) Spectral sequences.– A *spectral sequence of first kind* consists of (disjoint) chain complexes:

$$\text{For each } r \in \mathbb{Z}_{\geq 0}, \qquad \left(\{E_r^{p,q}\}_{p,q\in\mathbb{Z}_{\geq 0}}, D_r\right), \text{ where } D_r : E_r^{p,q} \to E^{p+r,q-r+1} \text{ is a differential.}$$

This sequence of complexes is assumed to be related by the following relation:

$$\boxed{E_{r+1}^{p,q} = \frac{\operatorname{Ker}(D_r : E_r^{p,q} \to E_r^{p+r,q-r+1})}{\operatorname{Im}(D_r : E^{p-r,q+r-1} \to E_r^{p,q})}}$$

Lemma. *For each $p, q \in \mathbb{Z}_{\geq 0}$, we have:*

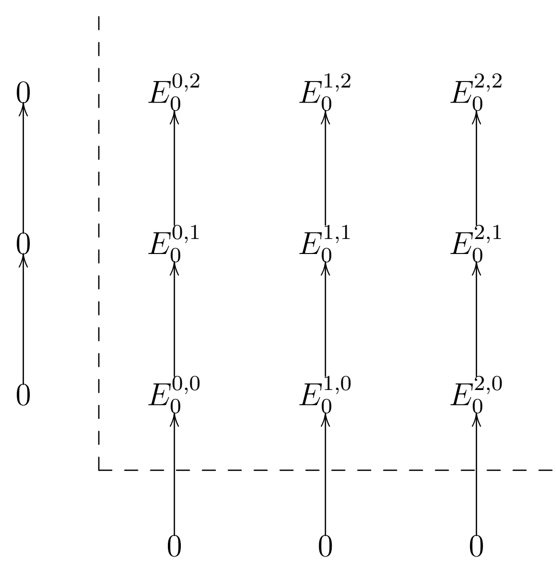
$$E_{r_1}^{p,q} = E_{r_2}^{p,q} \text{ for every } r_1, r_2 > \max(p, q+1).$$

We denote this object by $E_\infty^{p,q}$.

Proof. If $r > q+1$, then $D_r : E_r^{p,q} \to E^{p+r,q-r+1} = 0$, and if $r > p$, then $D_r : 0 = E^{p-r,q+r-1} \to E_r^{p,q}$, proving that $E_{r+1}^{p,q} = E_r^{p,q}$. This proves the lemma. □

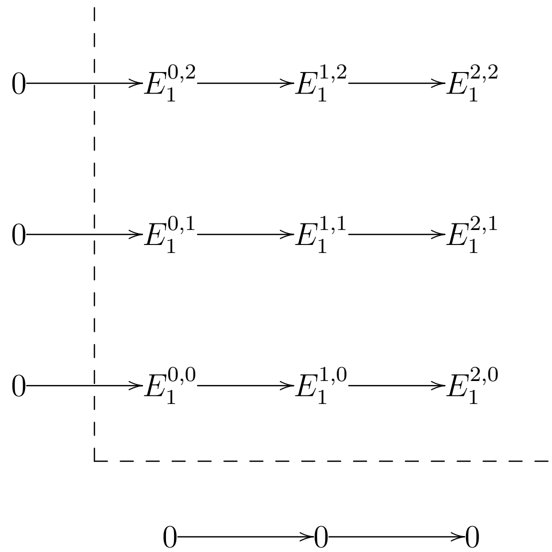
Example. For a fixed r, the chain complex $\{E_r^{p,q}, D_r\}_{p,q\in\mathbb{Z}_{\geq 0}}$ is often referred to as r^{th} page of the spectral sequence.

$r = 0$. In this case the differential is vertical. Here is the picture of the 0^{th} page:

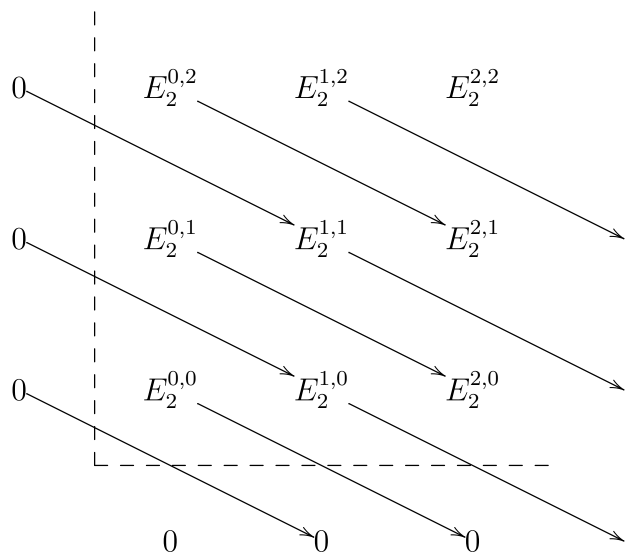


$r = 1$. Now the differential goes $E_1^{p,q} \to E_1^{p+1,q}$, i.e, hortizontally. Recall that

$$E_1^{p,q} = \frac{\operatorname{Ker}(E_0^{p,q} \to E_0^{p,q+1})}{\operatorname{Im}(E_0^{p,q-1} \to E_0^{p,q})}.$$



$r = 2$. Now $D_2 : E_2^{p,q} \to E_2^{p+2,q-1}$. Again, $E_2^{p,q} = \dfrac{\mathrm{Ker}(E_1^{p,q} \to E_1^{p+1,q})}{\mathrm{Im}(E_1^{p-1,q} \to E_1^{p,q})}$.



As r increases the direction of differentials keeps on turning clockwise: r steps to the right and $r - 1$ steps down.

A spectral sequence of second kind is defined analoguously, the only difference being that the differentials D_r go north-west, instead of south-east. $D_r : E_r^{p,q} \to E_r^{p-r+1,q+r}$. Alternately, it is obtained from those of first kind by flipping $(p, q) \leftrightarrow (q, p)$.

(23.3) Spectral sequences associated to a bicomplex.– Let $\{K^{\bullet,\bullet}, d, \partial\}$ be a bicomplex as in §23.1.

Theorem.

(1) *There exists a spectral sequence $(\mathcal{I}, \mathcal{D})$ of first kind, uniquely determined by the following initial conditions:*
   - $\mathcal{I}_0^{p,q} = K^{p,q}$. $\mathcal{D}_0 = \partial$ *(vertical differential).*

- $\mathcal{I}_1^{p,q} = H_\partial^q(K^{p,\bullet})$. $\mathcal{D}_1 = d$ *(horizontal differential). See the paragraph below for more details.*

(2) *There exists a spectral sequence* $(\mathbb{I}, \mathbb{D})$ *of second kind, uniquely determined by the following initial conditions:*
- $\mathbb{I}_0^{p,q} = K^{p,q}$. $\mathbb{D}_0 = d$ *(horizontal differential).*
- $\mathbb{I}_1^{p,q} = H_d^p(K^{\bullet,q})$. $\mathbb{D}_1 = \partial$ *(vertical differential).*

(3) *For fixed* $n \in \mathbb{Z}$, *we have:*

$$\boxed{H^n(K^{[\bullet]}) = \bigoplus_{p+q=n} \mathcal{I}_\infty^{p,q} = \bigoplus_{p+q=n} \mathbb{I}_\infty^{p,q}}$$

Remark. A few words of explanation are in order. In (1), $\mathcal{I}_1^{p,q}$ is the q^{th} cohomology of the following complex:

$$K^{p,\bullet}: \qquad 0 \to K^{p,0} \xrightarrow{\partial} K^{p,1} \xrightarrow{\partial} K^{p,2} \xrightarrow{\partial} \cdots$$

The horizontal differential d can be viewed as a morphism of complexes, $d : K^{p,\bullet} \to K^{p+1,\bullet}$, and hence induces a morphism at the level of cohomology, again denoted by d:

$$\mathcal{I}_1^{p,q} = H_\partial^q(K^{p,\bullet}) \xrightarrow{d} H_\partial^q(K^{p+1,\bullet}) = \mathcal{I}_1^{p+1,q},$$

which is what we are defining to be $\mathcal{D}_1$. Note that $\mathcal{I}_2^{p,q} = H_d^p(H_\partial^q(K^{\bullet,\bullet}))$.

Similarly $\mathbb{I}_1^{p,q} = H_d^p(K^{\bullet,q})$ with $\mathbb{D}_1$ induced from ∂ viewed as a morphism of complexes: $\partial : K^{\bullet,q} \to K^{\bullet,q+1}$. Thus,

$$\mathbb{I}_2^{p,q} = H_\partial^q(H_d^p(K^{\bullet,\bullet})).$$

I am not going to include a proof of this theorem here. It is not difficult, and only requires a careful diagram chase (see R.Vakil's notes cited above).

(23.4) Proof of Theorem 22.3.– Let $M, N \in R$-mod. Assume that we have projective resolution of M:

$$\cdots \xrightarrow{f_2} P_2 \xrightarrow{f_1} P_1 \xrightarrow{f_0} P_0 \to 0,$$

and an injective resolution of N:

$$0 \to I^0 \xrightarrow{g^0} I^1 \xrightarrow{g^1} I^2 \xrightarrow{g^2} \cdots$$

Form a bicomplex: $K^{p,q} = \mathrm{Hom}(P_p, I^q)$ with horizontal differential given by:

$$d : K^{p,q} = \mathrm{Hom}(P_p, I^q) \xrightarrow{-\circ f_p} \mathrm{Hom}(P_{p+1}, I^q) = K^{p+1,q}$$

and vertical differential by:

$$\partial : K^{p,q} = \mathrm{Hom}(P_p, I^q) \xrightarrow{g^q \circ -} \mathrm{Hom}(P_p, I^{q+1}) = K^{p,q+1}$$

By associativity of composition, we get $d\partial = \partial d$.

Spectral sequence of first kind $(\mathcal{I}, \mathcal{D})$. For $r = 0$, the complexes are aligned vertically. Thus for a fixed p, we have $\mathcal{D}_0 = \partial : K^{p,q} \to K^{p,q+1}$. This complex is nothing but $\mathrm{Hom}(P_p, I^\bullet)$. Since P_p is projective, its cohomology vanishes everywhere, except at 0^{th} spot, where we get $\mathrm{Hom}(P_p, N)$.

Thus at $r = 1$, we obtain the following answer:

$$\mathcal{I}_1^{p,q} = H_\partial^q(\mathrm{Hom}(P_p, I^\bullet)) = \begin{cases} \mathrm{Hom}(P_p, N) & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases}$$

The differential $\mathcal{D}_1$ at $r = 1$ step of the spectral sequence, is again $- \circ g_p : \mathcal{I}^{p,0} \to \mathcal{I}^{p+1,0}$. We conclude that $(\mathcal{I}^{\bullet,0}, \mathcal{D}_1)$ is same as $\mathrm{Hom}(P_\bullet, N)$.

At $r = 2$, we get $\mathcal{I}_2^{p,0} = H^p(\mathrm{Hom}(P_\bullet, N))$. Note that $\mathcal{D}_2 = 0$, which implies

$$\boxed{\mathcal{I}_\infty^{p,q} = \mathcal{I}_2^{p,q} = \begin{cases} H^p(\mathrm{Hom}(P_\bullet, N)) & \text{if } q = 0, \\ 0 & \text{if } q > 0. \end{cases}}$$

Carrying out the same analysis for the spectral sequence of second kind $(\mathbb{I}, \mathbb{D})$, we get:

$$\boxed{\mathbb{I}_\infty^{p,q} = \mathbb{I}_2^{p,q} = \begin{cases} H^q(\mathrm{Hom}(M, I_\bullet)) & \text{if } p = 0, \\ 0 & \text{if } p > 0. \end{cases}}$$

Theorem 23.3 (3) implies that the two cohomologies are the same.