Lecture 0

(0.0) Introduction. – The main object of study in Linear algebra is a system of linear equations. Such systems are conveniently encoded using matrices and are explicitly solved using an algorithm (Gauss-Jordon elimination).

In this course, we will go over this method in detail – this lecture and the next one are devoted to this. For the main part of this course, we will see a re-interpretation of matrices – by viewing them as linear transformations between vector spaces. The problem of solving a system of linear equations will also be re-phrased in the language of vectors and linear dependence. (We will go over definitions and examples of underlined words in the previous paragraph - don't worry!)

(0.1) Linear equations. A linear equation in n variables is an equation of the form (here $n = 1, 2, 3, \ldots$)

$$\boxed{a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b} \quad - (*)$$

- here $a_1, a_2, a_3, \ldots, a_n, b$ are fixed numbers (usually real numbers, or complex numbers).
- $x_1, x_2, \ldots, x_n$ are unknowns (variables).

The word "linear" refers to the fact that the <u>degree of each variable</u> on the left-hand side of (*) is 1.

Examples - (i) $2x_1 + 3x_2 = 6$ is a linear equation in two variables. $\boxed{\begin{aligned} x_1 &= 3 \\ x_2 &= 0 \end{aligned}}$ is a solution of this linear equation.

$\boxed{\begin{aligned} x_1 &= 0 \\ x_2 &= 2 \end{aligned}}$ is another solution.

(ii) $x + 2y + z = 9$ is a linear equation in three variables x, y and z.

(for $n = 2$, we are used to calling the variables x and y instead of x_1 and x_2. Similarly for $n = 3$ we are used to x, y, z (instead of x_1, x_2, x_3).)

(iii) $\sin(x_1) + \cos(x_2) = 1$ is <u>not</u> a linear equation

(0.2) A <u>system of linear equations</u> (or just <u>linear system</u>) is a finite collection of linear equations.

For example. - (I) $\left.\begin{aligned} x + y &= 3 \\ 2x - y &= 3 \end{aligned}\right\}$ - linear system of 2 equations in 2 variables.

(II) $\left.\begin{array}{l} x + y + 2z = 4 \\ 2x + y \quad\quad = 0 \end{array}\right\}$ linear system of 2 equations in 3 variables.

Goal: Find solutions to a linear system of equations.

Idea: Eliminate variables to get an equation with only one variable ~~on~~ in its left-hand side.

Example - (I) $\begin{array}{l} x + y = 3 \\ 2x - y = 3 \end{array}$ Adding the two equations gives $3x = 6 \Rightarrow \boxed{x = 2}$ Plug it back in the original equations to get $\boxed{y = 1}$.

Thus we conclude that the system (I) has <u>a unique solution</u>, namely $\boxed{\begin{array}{l} x = 2 \\ y = 1 \end{array}}$.

Example - (II) $\begin{array}{l} x + y + 2z = 4 \\ 2x + y \quad\quad = 0 \end{array} \rightsquigarrow \boxed{y = -2x}$

Replace y by $-2x$ in the first equation to get

$$-x + 2z = 4 \quad \Rightarrow \quad \boxed{x = 2z - 4}.$$

$$y = -2x = -4z + 8.$$

So, for any value of $z = t \in \mathbb{R}$ (real number), we have a solution : $\boxed{\begin{aligned} x &= 2t-4 \\ y &= -4t+8 \\ z &= t \end{aligned}}$. We see in this example that there are <u>infinitely many solutions of</u> (II).

Example (III) $\begin{aligned} x + 3y &= 4 \\ 2x+6y &= 10 \end{aligned}$ $\xrightarrow[\text{by } 2]{\text{Multiply the } 1^{st} \text{ equation}}$ $\begin{aligned} 2x+6y &= 8 \\ 2x+6y &= 10 \end{aligned}$

Subtracting, we get $0 = 2$! What we conclude here is that the system (III) has <u>no solutions</u> at all.

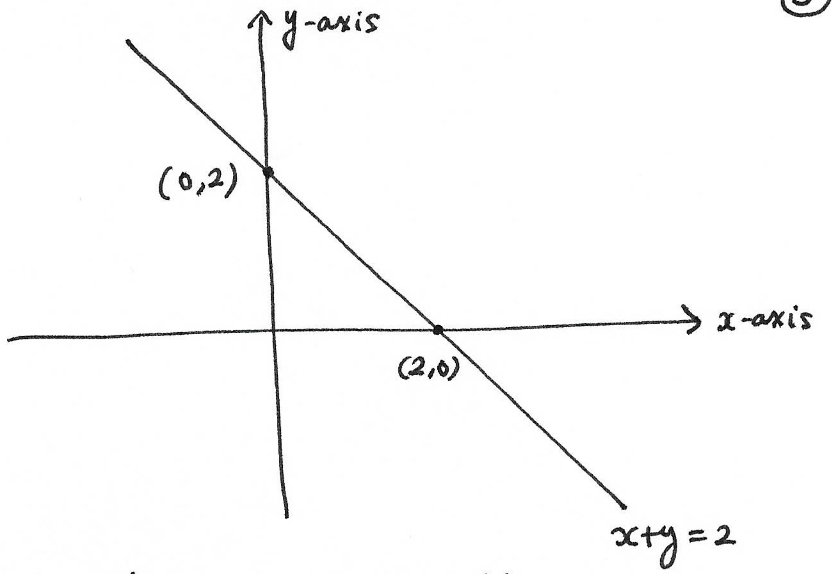
(0.3) Later in the course, we will see that:

> Every linear system of equations has either
> 0, 1 or ∞ number of solutions
> infinity

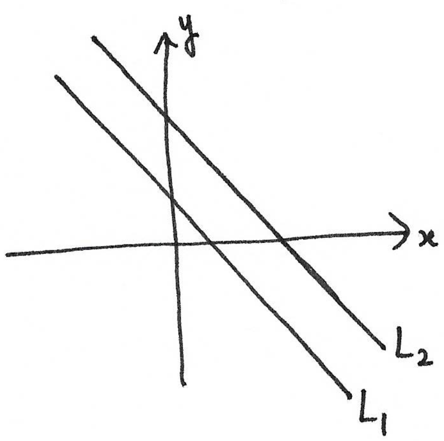
For $n = 2$ or 3 (number of variables is 2 or 3) we can see geometrically why this is true.

(0.4) <u>$n=2$</u>. Every linear equation in 2 variables defines a line in the plane $\mathbb{R}^2$.

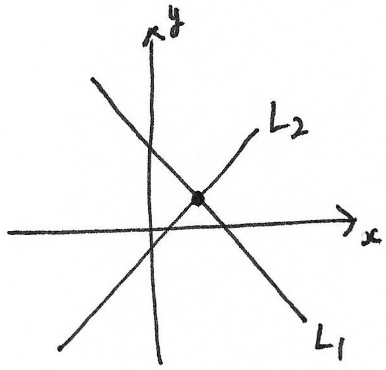
$x + y = 2 \quad \rightsquigarrow$



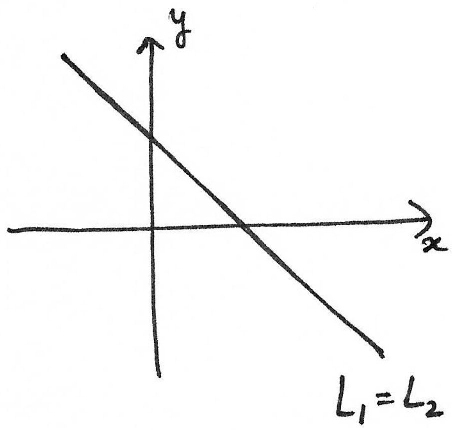
Assume we are given <u>2 linear equations in 2 variables</u>. Geometrically, the two lines L_1 and L_2 defined by these equations are either parallel, or intersect at a unique point, or $L_1 = L_2$



No solutions



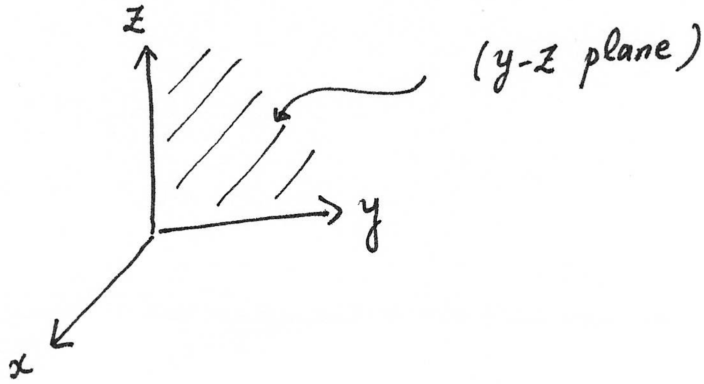
Unique solution



Infinitely many solutions

(0.5) <u>$n = 3$</u>. Every linear equation in 3 variables defines a <u>plane</u> in $\mathbb{R}^3$.

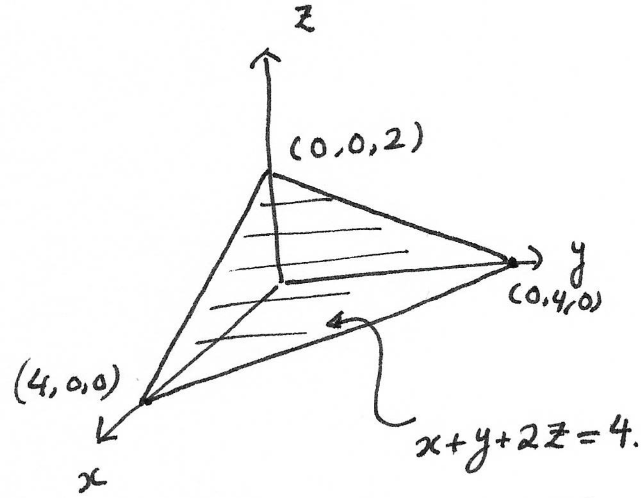
e.g. $x = 0 \quad \rightsquigarrow$



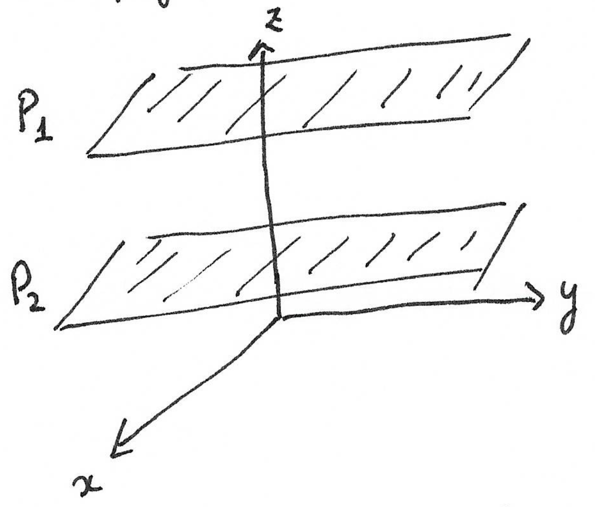
- If the linear system consists of only one equation, then there are infinitely many solutions (as many as the points on the plane it defines).

e.g. $x + y + 2z = 4$. For any choice of $\left.\begin{matrix} x = s \\ y = t \end{matrix}\right\}$ real numbers

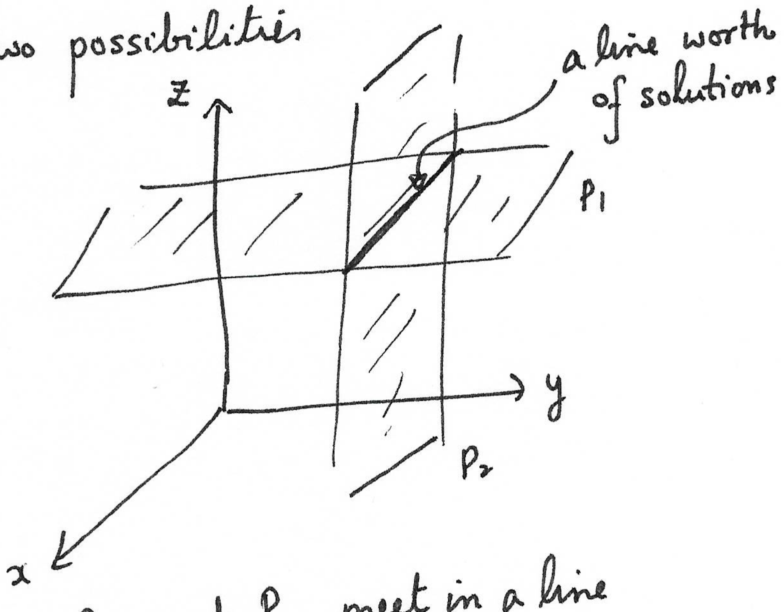
we have a solution
$$\boxed{\begin{aligned} x &= s \\ y &= t \\ z &= \frac{4-s-t}{2} \end{aligned}}$$



- If the linear system consists of 2 equations (see Example (II) on page 3 above), there are two possibilities



Two planes are parallel
(No solution)



P_1 and P_2 meet in a line
(again ∞ number of solutions)

e.g. system (II) from page 3: $\left\{ \begin{aligned} x+y+2z &= 4 \\ 2x+y \quad &= 0 \end{aligned} \right\}$

Solutions of this system can be identified with the following line :

$$\left.\begin{aligned} x &= 2t-4 \\ y &= -4t+8 \\ z &= t \end{aligned}\right] \leftarrow$$

parametric form of the line through $\underbrace{(-4,8,0)}$ set $t=0$ along the direction vector $\underbrace{\langle 2,-4,1\rangle}_{\text{coefficients of } t}$

Remarks.- (i) Later in the course, we will see a proof of the fact that if $\boxed{\#\text{ equations} < \#\text{ variables}}$ then there are either no solutions, or infinitely many solutions.

(ii) ~~Even~~ When a system admits infinitely many solutions, the set of solutions can be <u>parametrized</u> by a certain <u>number of free parameters</u> - we will also discuss the importance of this number in what will be called "rank-nullity theorem".

(e.g. last page: $x+y+2z=4$
Solutions are described using 2 free parameters)