(3.0) Summary so far.

- $\boxed{\text{Linear Systems}} \rightsquigarrow \boxed{\text{Matrices}}$

e.g. a system of m linear equations in n variables

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}} \longleftrightarrow \boxed{\begin{array}{c} m \times (n+1) \text{ matrix} \\ B = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \end{array}}$$

Coefficients — column of right-hand side.

- <u>Elementary row operations</u> on matrices <u>do not change the set of solutions</u> of the associated linear system

(Row operations:
- Swap : $R_i \leftrightarrow R_j$
- Scale : $R_i \to \alpha R_i \quad (\alpha \neq 0)$
- Combine : $R_i \to R_i + \lambda R_j \quad (j \neq i;\ \lambda$ arbitrary number$)$)

- Any matrix can be put in a <u>reduced echelon form</u> using these 3 operations.

- A linear system, <u>whose augmented matrix is in reduced echelon form</u>, is <u>already SOLVED</u>.

(3.1) Echelon vs reduced echelon form.

<u>Echelon form</u> is easily recognized from 3 properties:

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & * & * & \cdots & * \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 & * \cdots & * \\ & & \vdots & & & & & & \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 * \cdots * \\ \hline 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & & \vdots & & & & & & \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \end{bmatrix}$$

(2) Leading term of a non-zero row is 1.

(3) Staircase shape. <u>Leading 1</u> of a row is to the right of the leading 1 of the previous row.

(1) all zero rows are grouped at the bottom.

If, in addition, the <u>leading</u> (or <u>pivotal, or corner</u>) 1 is the <u>only non-zero entry of its column</u>, the matrix is said to be in a <u>reduced echelon form</u>.

e.g. $\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 7 \end{bmatrix}$ is in reduced echelon form

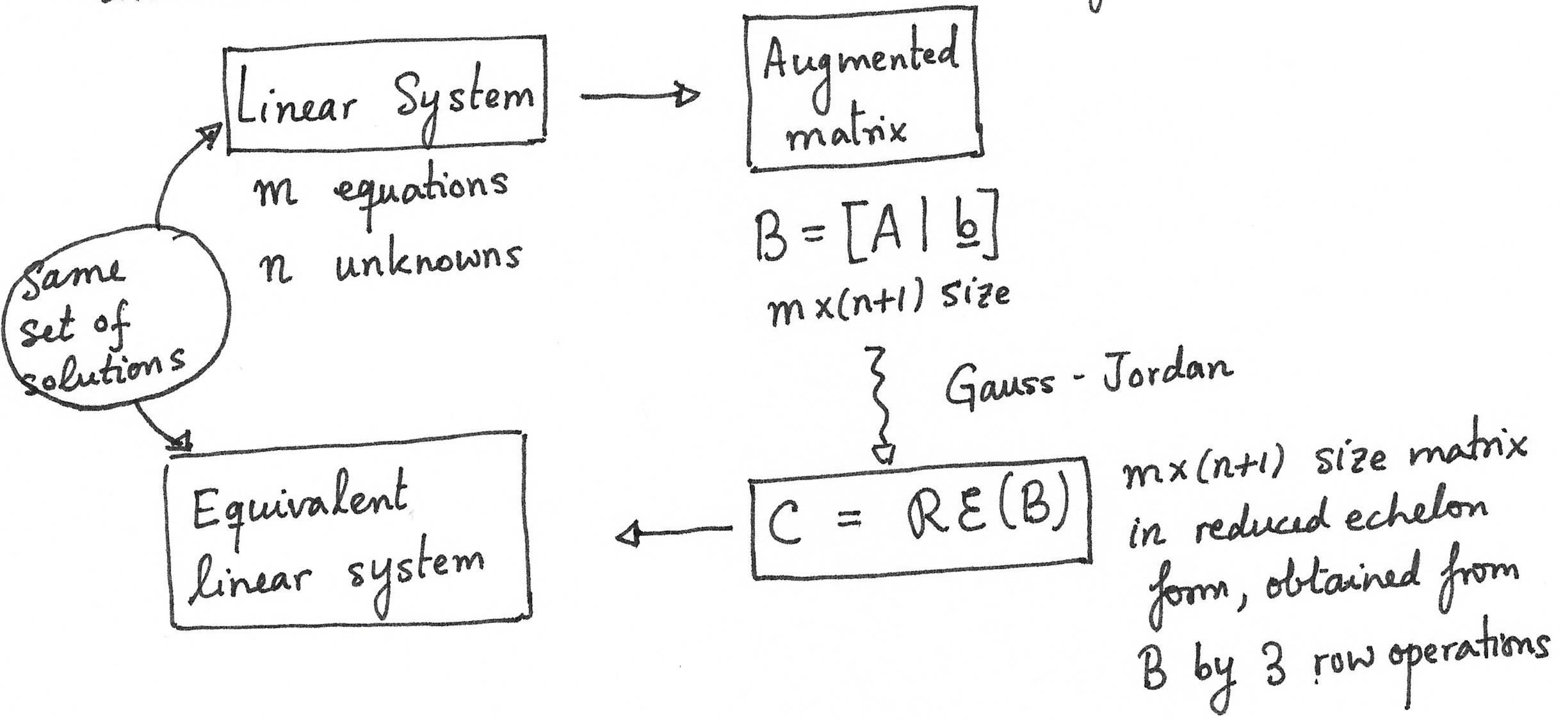
$\begin{bmatrix} 1 & 0 & 5 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is <u>NOT</u> in reduced echelon form but only in an echelon form.

(this leading 1 is <u>NOT</u> the only non-zero entry of its column = column #4)

Last lecture: algorithm to put a matrix in reduced echelon form, using the three elementary row operations.

(3.2) Linear system whose augmented matrix is in a reduced echelon form.

Last time we made some observations about the linear system associated to a matrix in reduced echelon form.



- If $C = RE(B)$ contains the row $[0\ 0 \cdots 0 \mid 1]$, then the linear system under consideration is <u>inconsistent</u> (i.e., set of solutions $= \phi$: empty set.)
- Assume the system is <u>consistent</u>. Locate the <u>leading 1's</u> in $C = RE(B)$. The variables corresponding to <u>column numbers of leading 1's</u> are called <u>dependent</u> variables, and the rest are called <u>independent variables</u> or <u>free parameters</u>

  \# Independent variables $= 0$ ⇝ Unique solution

  \# Independent variables > 0 ⇝ Infinitely many solutions.

e.g. Let us take a matrix in reduced echelon form and see whether the corresponding system has 0, 1 or ∞ number of solutions

(i) $\left[\begin{array}{cccc|c} 1 & 3 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 9 \\ \hline 0 & 0 & 0 & 0 & 1 \end{array}\right]$ appearance of such a row means <u>NO solutions</u>

(ii) $\begin{array}{c} \left[\begin{array}{ccccc|c} 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccccc} & 2^{nd} & & 4^{th} & 5^{th} & \end{array} \end{array}$

- Consistent ✓
- 3 dependent variables x_2, x_4, x_5
- 2 independent variables x_1, x_3.

So, <u>infinitely many solutions</u> :

for any choice of 2 numbers $x_1 = s$, $x_3 = t$, we have a solution :

Row 3 $\rightsquigarrow$ $x_5 = 2$
Row 2 $\rightsquigarrow$ $x_4 = 3$
Row 1 $\rightsquigarrow$ $x_2 = x_3 = t$ and $x_1 = s$.

(iii) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & 12 \end{array}\right]$

- consistent ✓
- all 3 variables are dependent.
- # independent variables = 0

Hence, <u>unique solution</u> : $x_1 = 5$
$x_2 = 13$
$x_3 = 12$

Thus it should be clear what we mean when we say that a linear system, whose <u>augmented matrix is in reduced echelon form</u>, is <u>already solved</u>.

(3.3) Consequences. – (1) A linear system admits $\underline{0, 1 \text{ or } \infty}$ number of solutions.

(2). If # equations < # variables, the linear system <u>cannot have a unique solution.</u>

<u>Reason.</u> – (1): Assume m = number of equations
n = number of variables

$$B = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$ is the augmented matrix of given linear system.

$C = RE(B)$ matrix in reduced echelon form obtained from B via 3 elementary operations.

→ If C contains $[0\ 0 \cdots 0 \,|\, 1]$, number of solutions $= 0$.

→ If C does not contain $[0\ 0 \cdots 0 \,|\, 1]$ (i.e. <u>system is consistent</u>),

let r = # of non-zero rows in C.

Then (a) $\boxed{r \leq m}$. (since m = total number of rows.)

(b) $\boxed{r \leq n.}$ This is a little non-trivial. We can easily see that $r \leq$ total number of columns $= n+1$.

Because every non-zero row in C has a leading 1, which is the <u>only non-zero entry</u> of its column.

If $r = n+1$, every column has to contain a leading 1, including the last ($n+1^{st}$) column. But if that were the case, C would contain $[0\ 0 \cdots 0 \,|\, 1]$ and be <u>inconsistent</u>.

Now there are only 2 possibilities: $r = n \rightsquigarrow$ unique solution (no independent variables)

$r < n \rightsquigarrow$ infinitely many solutions, parametrized by $n - r$ free/independent variables.

(2) If $m < n$, we see that either the system is inconsistent, or $r \leq m < n \rightsquigarrow \underline{n - r > 0}$ i.e., # Independent variables > 0. Hence, infinitely many solutions.

(3.4) <u>Homogeneous Systems.</u> A linear system of the general form

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}}$$

is said to be <u>homogeneous</u> if $b_1 = b_2 = \dots = b_m = 0$.

A homogeneous system <u>is always consistent</u>, since $\boxed{\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ &\vdots \\ x_n &= 0 \end{aligned}}$ is certainly a solution.

(often called <u>the trivial solution</u>).

Homogeneous System $\rightarrow$ Unique solution (i.e., trivial solution is the only solution)

or

Homogeneous System $\rightarrow$ Infinitely many Solutions (we say "<u>the homogeneous system has non-trivial solutions</u>")

e.g. $\boxed{\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ 2x_1 + x_3 + x_4 &= 0 \\ x_2 - 2x_3 - x_4 &= 0 \end{aligned}}$ ← homogeneous linear system with 3 equations and 4 variables.

- System is consistent, since it is homogeneous. Moreover, $m < n$ here, so there must be <u>infinitely many solutions</u>.

Explicitly, $B = \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 \end{array}\right]$. Let us get it into a reduced echelon form.

$$\xrightarrow[R_2 \to R_2 - 2R_1]{} \left[\begin{array}{ccccc} 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 \end{array}\right] \xrightarrow[R_2 \to \frac{1}{2}R_2]{} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_3 \to R_3 - R_2]{} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{-3}{2} & \frac{-3}{2} & 0 \end{array}\right] \xrightarrow[R_3 \to \frac{-2}{3}R_3]{} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & \frac{-1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

negative one

$$\xrightarrow[R_2 \to R_2 + \frac{1}{2}R_3]{} \left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow[R_1 \to R_1 - R_3]{} \left[\begin{array}{cccc|c} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 \to R_1 + R_2]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

x_4 is an independent variable

$$\boxed{\begin{aligned} x_4 &= t \text{ arbitrary} \\ x_3 &= -x_4 = -t \\ x_2 &= -x_4 = -t \\ x_1 &= 0 \end{aligned}}$$