Lecture 4

(4.0) Recall - last time we concluded our study of linear systems.

Given		Construct		Compute
Linear system of m equations in n variables	$\rightsquigarrow$	$m \times (n+1)$ size matrix B (augmented matrix of the linear system)	$\rightsquigarrow$	$G = RE(B)$ reduced echelon form.

Computationally long process

• Does G contain $[0\ 0\ \cdots\ 0 \mid 1]$?

Yes: The system is inconsistent. Set of solutions = empty

No:
- • define r = # of non-zero rows of G (rank)

  We proved that $\boxed{r \leq m \ \&\ r \leq n}$.
- • There are $n-r$ free / independent variables and r dependent variables (corresponding to columns containing leading 1's).

$n-r=0$: The system has a unique solution

$n-r>0$: The system has infinitely many solutions

A <u>homogeneous system</u> is ~~th~~ a linear system whose right-hand side consists entirely of 0's. That is, the last column of its augmented matrix is $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$. Such a system is always <u>consistent</u>.

Homogeneous linear system → # Solutions = 1

or

→ # Solutions = ∞

($x_1 = x_2 = \dots = x_n = 0$ is a solution, called the trivial solution)

(We say that the system admits non-trivial solutions).

(4.1) An application of homogeneous systems† (see Problem #50 of §1.2)

Finding a conic that passes through 5 prescribed points on 2-dimensional plane.

The general equation defining a conic (in $\mathbb{R}^2$) is

$$\boxed{Ax^2 + By^2 + Cxy + Dx + Ey + F = 0} \quad - (*)$$

e.g. Circle : $x^2 + y^2 = 4$. Ellipse : $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Parabola : $y - x^2 = 0$. Hyperbola : $x^2 - y^2 = 1$.

Idea : Plug-in the 5 prescribed points into (*) - treat A, B, C, D, E, F as unknowns $\rightsquigarrow$ 5 x 6 homogeneous system.

It must have some non-trivial solution (why?).

For example : let us find a conic that passes through

(-1, 2) (-4, 1) (3, 2) (5, 1) (7, -1)

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† Optional.

- Substitute $x=-1, y=2$ in (*) to get

	$A + 4B - 2C - D + 2E + F = 0.$	– (eq. 1)
• $x=-4$, $y=1$	$16A + B - 4C - 4D + E + F = 0$	– (eq. 2)
• $x=3$, $y=2$	$9A + 4B + 6C + 3D + 2E + F = 0$	– (eq. 3)
• $x=5$, $y=1$	$25A + B + 5C + 5D + E + F = 0$	– (eq. 4)
• $x=7$, $y=-1$	$49A + B - 7C + 7D - E + F = 0$	– (eq. 5)

(I solved this by hand - and found it more convenient to list the variables in reverse order
$\underset{x_1}{\overset{\shortparallel}{F}}, \underset{x_2}{\overset{\shortparallel}{E}}, \underset{x_3}{\overset{\shortparallel}{D}}, \underset{x_4}{\overset{\shortparallel}{C}}, \underset{x_5}{\overset{\shortparallel}{B}}$ and $\underset{x_6}{\overset{\shortparallel}{A}}$).

<u>Coefficient Matrix</u>

$$\begin{bmatrix} 1 & 2 & -1 & -2 & 4 & 1 \\ 1 & 1 & -4 & -4 & 1 & 16 \\ 1 & 2 & 3 & 6 & 4 & 9 \\ 1 & 1 & 5 & 5 & 1 & 25 \\ 1 & -1 & 7 & -7 & 1 & 49 \end{bmatrix} \xrightarrow{RE(\cdot)} \begin{array}{c} \begin{array}{cccccc} F & E & D & C & B & A \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{113}{3} \\ 0 & 1 & 0 & 0 & 0 & -18 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \end{bmatrix} \end{array}$$

(people will use a computer for this)

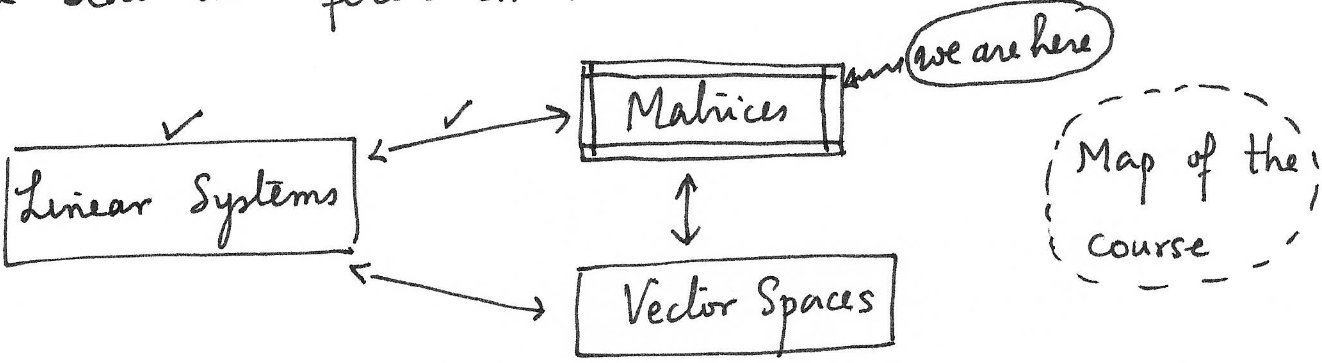
A is a free parameter. and $B = \frac{-1}{3}A$; $C = -A$; $D = 0$; $E = 18A$;
$F = \frac{-113}{3}A$.

$\boxed{\text{Set } A = 3}$ (just to get rid of the denominator)

$$\boxed{3x^2 - y^2 - 3xy + 54y - 113 = 0}$$

this is a hyperbola.

(4.2) Now we are going to put linear systems on back seat and focus on matrices.



Recall that an $m \times n$ matrix is a rectangular array of numbers, with m rows and n columns.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Short hand notation:

$$A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$$

Or, just $A = (a_{ij})$ if the size $m \times n$ is understood.

Definition. – Two matrices are equal when they have the same size and same entries. More explicitly:

if $A = (a_{ij})_{\substack{1 \le i \le m \\ 1 \le j \le n}}$ and $B = (b_{ij})_{\substack{1 \le i \le r \\ 1 \le j \le s}}$ are two matrices of sizes $m \times n$ and $r \times s$ respectively, then

$\boxed{A = B}$ means

$m = r$, $n = s$, and for every (i, j) with $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$ $a_{ij} = b_{ij}$

e.g. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 7 \end{bmatrix}$ (different sizes)

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ (same size, but (1,1)-entries are different)

(1,2) - " " "
(2,1) - " " " also.

There are 3 main operations we can perform on matrices.

- Addition
- Scalar multiplication
- Multiplication.

(4.3) Addition of two matrices. - Given two matrices of <u>same size</u>, say $A = (a_{ij})$ $B = (b_{ij})$ both $m \times n$ matrices, $A + B$ is defined to be an $m \times n$ matrix whose $(i,j)^{th}$ entry is $a_{ij} + b_{ij}$.

$$\boxed{(A+B)_{ij} = a_{ij} + b_{ij}}$$

<u>Note</u>: Matrix addition is <u>NOT DEFINED</u> for matrices <u>of different sizes</u>.

e.g.
$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 7 & 4 \\ 0 & -4 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 9 & 3 \\ 2 & -1 & 2 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \end{bmatrix}$ <u>NOT</u> defined.

(4.4) $\boxed{\text{Scalar}}$ multiplication. Given an $m \times n$ size matrix

just another word for a number - real or complex

$A = (a_{ij})$, and a number λ,

λA = $m \times n$ size matrix whose $(i,j)^{th}$ entry is λa_{ij}.

$$\boxed{(\lambda A)_{ij} = \lambda a_{ij}}$$

e.g. $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \\ 0 & 4 \end{bmatrix} \rightsquigarrow 5A = \begin{bmatrix} 15 & -10 \\ 5 & 0 \\ 0 & 20 \end{bmatrix}$.

$B = \begin{bmatrix} 1 & 9 \\ -2 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 6 & 0 \\ 3 & 10 \end{bmatrix} \rightsquigarrow 2B - C = 2B + (-1)C$

$$= \begin{bmatrix} 2 & 18 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} -6 & 0 \\ -3 & -10 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 18 \\ -7 & -2 \end{bmatrix}.$$

(4.5) Matrix multiplication.

Given $\boxed{\begin{array}{lll} A = (a_{ij}) & m \times n & \text{matrix} \\ B = (b_{kl}) & r \times s & \text{matrix} \end{array}}$ the product $A \cdot B$

is defined only when $\boxed{n = r}$; i.e.

Columns of A = # Rows of B.

Assuming this is the case $(n = r)$, the product $A \cdot B$ is an $m \times s$ sized matrix

$$\boxed{\begin{matrix} A \\ m \times n \end{matrix}} \quad \boxed{\begin{matrix} B \\ n \times s \end{matrix}} \quad \rightsquigarrow \quad \boxed{\begin{matrix} AB \\ m \times s \end{matrix}}$$

must match; or else the product <u>is NOT defined</u>.

If $\begin{matrix} i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, s \end{matrix}$; $(i,j)^{th}$ entry of $A \cdot B$ is defined as:

$$\boxed{(AB)_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}}$$

Summation notation $\displaystyle\sum_{k=1}^{n} a_{ik} b_{kj}$

Pictorially: to get $(i,j)^{th}$ entry of the product AB - focus on $\boxed{i^{th} \text{ row of } A}$ and $\boxed{j^{th} \text{ column of } B}$

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \quad \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} \rightsquigarrow$$

multiply corresponding entries; and add them up

$$\rightsquigarrow \underbrace{a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}}$$

$(i,j)^{th}$ entry of $A \cdot B$

(4.6) In the next lecture, we will see why this is the "right definition", but for now, let us see some examples.

(i) $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ NOT defined.

2×3 $\quad$ 2×2

must match!

(ii) $\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 7 \\ -2 & 4 & 5 \end{bmatrix}$ is defined and the

$\underline{\underline{2 \times 3}}$ ✓ $\underline{\underline{3}} \times 3$

result is 2×3 - matrix.

(1,1) entry of the product: $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \rightsquigarrow 1(1) + 0(0) + 3(-2) = -5.$

(1,2) entry of the product: $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} \rightsquigarrow 1(-1) + 0(2) + 3(4) = 11$

(1,3) entry of the product: $\begin{bmatrix} 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 7 \\ 5 \end{bmatrix} \rightsquigarrow 1(3) + 0(7) + 3(5) = 18$

---- compute (2,1), (2,2) and (2,3) entries ----

$$\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 7 \\ -2 & 4 & 5 \end{bmatrix} = \begin{bmatrix} -5 & 11 & 18 \\ -1 & 7 & 22 \end{bmatrix}$$

$2 \times 3 \cdot 3 \times 3$ ✓ $\quad 2 \times 3$