Lecture 7

(7.0) Recall that list time we listed algebraic properties of matrix operations - akin to those of ordinary numbers. I mentioned those which hold for numbers but no longer valid when we get to matrices. To state those precisely, we need the following matrix.

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$n \times n$ size called the <u>identity matrix</u>.

e.g. $I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. The identity matrix is a <u>square matrix</u>

<u>Square matrices</u> are those for which #Rows = # Columns.

(7.1) Identity matrices play the role of number 1, in matrix world.

<u>Proposition</u>. - Let A be an $m \times n$ matrix. Then

$$I_m \cdot A = A = A \cdot I_n.$$

Let us check this when A is $m \times 1$ matrix, i.e., a column vector

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{m \times m} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}}_{m \times 1} = \begin{bmatrix} u_1 + 0u_2 + 0u_3 + \ldots + 0u_m \\ 0u_1 + u_2 + 0u_3 + \ldots + 0u_m \\ \vdots \\ 0u_1 + 0u_2 + \ldots + 0u_{m-1} + u_m \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

In general if we write $A = \underbrace{\left[\vec{u}_1 \;\; \vec{u}_2 \;\; \cdots \;\; \vec{u}_n\right]}$

n column vectors ; each has m entries.

$$I_m A = \left[I_m \vec{u}_1 \;\; I_m \vec{u}_2 \;\; \cdots \;\; I_m \vec{u}_n \right]$$
$$= \left[\vec{u}_1 \;\; \vec{u}_2 \;\; \cdots \;\; \vec{u}_n \right] = A.$$

check by multiplying matrices

e.g. $A = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 9 \end{bmatrix}_{2\times 3}$; $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 9 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 3 & 7 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(7.2) Properties of <u>multiplication of numbers</u>, which <u>fail</u> <u>for matrices</u> :

(1) $ab = ba$ for numbers a, b ; $AB \neq BA$ for matrices A, B.

(we saw example of this last time).

(2) $ab = 0$ means either $a = 0$, or $b = 0$. ; AB could be 0 for non-zero matrices A, B.

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0_{2\times 2}$.

(3) $a \neq 0$ means we can always find $b \left(= \frac{1}{a}\right)$ such that $ab = 1$. ; There are non-zero matrices, say A, for which no such B can be found. (i.e. which will make $AB =$ identity matrix)

e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Let us try to find $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

so that $AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Now $AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$

Hence, no matter which 2×2 matrix we pick, $\neq 1$.

$AB \neq I_2$.

(7.3) Invertible matrices. An $(n\times n)$ matrix (i.e., square matrix) A is said to be invertible if there exists an $(n\times n)$ matrix B such that $AB = I_n = BA$. Such a matrix B is called "inverse of A". It is unique, and denoted by A^{-1}.

(Why is it unique? If B and B' are two $(n\times n)$ matrices such that $AB = I_n = BA$, and $AB' = I_n = B'A$, then:

$$\boxed{B} = B\cdot I_n = B\cdot(AB') \underset{\substack{\text{matrix multiplication} \\ \text{is associative}}}{=} (BA)B' \underset{\text{use } BA = I_n}{=} I_n B' = \boxed{B'}.)$$

e.g. • I_n is always invertible (for every n), since $I_n \cdot I_n = I_n$, $\boxed{I_n^{-1} = I_n}$ (just like $\frac{1}{1} = 1$)

• $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is invertible, with $A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Check: $AA^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & -1+1 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$.

Similarly $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 1-1 \\ 0+0 & 0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$.

(7.4) Significance of invertible matrices. - for linear systems

If
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= b_n \end{aligned}$$
is a linear system of $\underline{n}$ equations, $\underline{n}$ variables

written in matrix form $\boxed{A\vec{x} = \vec{b}}$ (here $A = (a_{ij})_{n\times n}$ is a square matrix).

and assume that A is invertible, then:

$\boxed{\vec{x} = A^{-1}\vec{b}}$ is the unique solution.

(*). $\boxed{A = (a_{ij}) \text{ } n\times n \text{ matrix is invertible} \Rightarrow \text{Every linear system } A\vec{x}=\vec{b} \text{ has a unique solution}}$.

This is the most important feature of invertible matrices, which in fact characterizes them, and allows us to compute A^{-1} using Gauss-Jordan algorithm. We are going to argue for the converse of this assertion (*), which leads to a method for figuring out what A^{-1} is.

(7.5) Assume $A = (a_{ij})$ is an $n \times n$ matrix such that the linear system $A\vec{x} = \vec{b}$ always has a unique solution.

(no matter which $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ we are given)

Take $\vec{b}$ to be $\underbrace{\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \dots, \ \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\text{all } n \times 1 \text{ matrices (i.e. in } \mathbb{R}^n)}$

called coordinate vectors

Since a unique solution is assumed to exist, we find n column vectors:

$$\vec{x}_1 = \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{bmatrix}, \quad \dots, \quad \vec{x}_n = \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{bmatrix}$$

unique soln. of $A\vec{x} = \vec{e}_1$ unique soln. of $A\vec{x} = \vec{e}_2$ unique soln. of $A\vec{x} = \vec{e}_n$

Then: $$\boxed{A^{-1} = \begin{bmatrix} x_{11} & x_{12} & & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & & x_{nn} \end{bmatrix}}.$$

Why? $$A\,[\vec{x}_1 \ \vec{x}_2 \ \cdots \ \vec{x}_n] = [A\vec{x}_1 \ A\vec{x}_2 \ \cdots \ A\vec{x}_n]$$

$$= [\vec{e}_1 \ \vec{e}_2 \ \dots \ \vec{e}_n] = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & 0 & \cdots & \vdots \\ & \vdots & & 0 \\ 0 & 0 & & 1 \end{bmatrix} = I_n.$$

Thus, if $X = \begin{bmatrix} x_{11} & x_{12} & & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & & x_{nn} \end{bmatrix}$ then $\boxed{AX = I_n}$.

(Note: we have not checked that $XA = I_n$ as well. Next time we will see that <u>this is automatic</u>, but for now let us just focus on $AX = I_n$.)

(7.6) Reduced echelon form. - If $A = (a_{ij})$ is an $n \times n$ matrix such that every linear system $\boxed{A\vec{x} = \vec{b}}$ has <u>unique solution</u>, then let $\boxed{G = RE(A)}$ (reduced echelon form of A - obtained by applying Gauss-Jordan elimination algorithm - see Lecture 2 page 4).

<u>Proposition</u>. - $G = I_n$.

<u>Proof</u>. - Let $r = \#$ of non-zero rows of G (rank of the system).
If <u>$r < n$</u>, then we will have <u>$n - r > 0$ free parameters</u>, meaning there will be <u>infinitely many solutions</u>.
But we are assuming $\#$ Solns $= 1$.

So, $r = n$. Hence,
$$G = \begin{bmatrix} \boxed{1} & & & & & 0 \\ & \boxed{1} & & & & \\ & & \boxed{1} & & & \\ & & & \boxed{1} & & \\ & & & & \ddots & \\ 0 & & & & & \boxed{1} \end{bmatrix} = I_n.$$
(since $r = n$, staircase cannot skip a step)

In conclusion, using $(*)$ from page 4 above,

$$\boxed{A = (a_{ij}) \text{ invertible } \Rightarrow RE(A) = I_n}$$

So, <u>all invertible matrices</u> are <u>row equivalent to I_n</u>.

(7.7) Method to compute A^{-1} ($A = (a_{ij})$: $n \times n$, invertible).

(i) Form an $(n \times 2n)$ matrix $[A \mid I_n]$

(ii) Use Gauss-Jordan algorithm to get A into reduced echelon form

$$[A \mid I_n] \dashrightarrow [\underbrace{I_n}_{RE(A) = I_n} \mid B]$$

Conclude: $A^{-1} = B$.

<u>Remark.</u> $[A \mid I_n]$ is the augmented matrix of n linear systems $A\vec{x_1} = \vec{e_1}$, $A\vec{x} = \vec{e_2}$, ..., $A\vec{x} = \vec{e_n}$ (see page 5)

This method is basically <u>solving all these systems in one go</u>.

e.g. $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Let us compute A^{-1}.

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 5 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \to R_2 - 2R_1} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_2 \to (-1)R_2]{} \underbrace{\left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]}_{\text{echelon form NOT reduced}} \xrightarrow[R_2 \to R_2 - R_3]{} \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \to R_1 - R_3} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1 \to R_1 - 3R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & 2 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

reduced echelon form $= I_3$

$$A^{-1} = \begin{bmatrix} -5 & 3 & 2 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Check: $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -5 & 3 & 2 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

$$= \begin{bmatrix} -5 & 3 & 2 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

How did we know A was invertible?

Ans. - We didn't. So far, our only test for invertibility is:

$$A \rightsquigarrow RE(A) = I_n? \quad \xrightarrow{\text{Yes}} \text{Invertible} \qquad \xrightarrow[\text{No}]{} \text{Not invertible.}$$

Later we will define determinants; $\det(A)$ will be a number:

$\det(A) = 0 \longleftrightarrow$ Not invertible.

$\det(A) \neq 0 \longleftrightarrow$ Invertible.

(7.8) Algebraic properties of matrix inverses.

Let A and C be two $n \times n$ invertible matrices.

Theorem. - (i) $I_n^{-1} = I_n$ (since $I_n \cdot I_n = I_n$).

(ii) $(A^{-1})^{-1} = A$ (since $A(A^{-1}) = I_n = (A^{-1})A$; A is the inverse of A^{-1}!)

(iii) $(AC)^{-1} = C^{-1}A^{-1}$ (since $(AC)(C^{-1}A^{-1}) = A\underline{CC^{-1}}A^{-1} = AA^{-1} = I_n$.)

(iv) $(A^T)^{-1} = (A^{-1})^T$ (see last lecture for transpose) (since $AA^{-1} = I_n \Rightarrow (AA^{-1})^T = I_n^T = I_n$ i.e., $(A^{-1})^T A^T = I_n$. So $(A^{-1})^T$ is the inverse of A^T.)

(v) ($\alpha \neq 0$ number) $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$ (since $(\alpha A)(\frac{1}{\alpha}A^{-1}) = AA^{-1} = I_n$).