Lecture 9

(9.0) Recall - last time we introduced the notion of <u>linearly independent</u> and <u>dependent</u> set of vectors:

Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be n vectors in $\mathbb{R}^m$ (n, m : two positive integers)

- Any expression of the form $x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n$ is called (where $x_1, x_2, \ldots, x_n \in \mathbb{R}$) a <u>linear combination</u> of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$.

- The equation $\boxed{x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_n\vec{v}_n = \vec{0}}$ is same as a <u>homogeneous</u> linear system of <u>m equations</u> in <u>n variables</u>.

  If $x_1 = x_2 = \ldots = x_n = 0$ is <u>the only solution</u> of this homogeneous system (i.e., trivial solution), we say $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are <u>linearly independent</u>

  If the homogeneous system admits a <u>non-trivial</u> solution, we say $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are <u>linearly dependent</u>.

- Assuming $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are <u>linearly dependent</u>, say

  $x_1 = \alpha_1, x_2 = \alpha_2, \ldots, x_n = \alpha_n$ is a non-trivial solution.

  That is, <u>not all</u> <u>α_i's are zero</u> and

  $$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \ldots + \alpha_n\vec{v}_n = \vec{0}.$$

  Let us assume $\alpha_1 \neq 0$. Then $\boxed{\vec{v}_1 = -\frac{\alpha_2}{\alpha_1}\vec{v}_2 + \left(-\frac{\alpha_3}{\alpha_1}\right)\vec{v}_3 + \ldots + \left(-\frac{\alpha_n}{\alpha_1}\right)\vec{v}_n}$

  In words we say that <u>this</u> expresses <u>$\vec{v}_1$ as a linear combination of</u>

$\vec{v}_2, \vec{v}_3, \dots, \vec{v}_n$.

(9.1) Examples. – (i) Let $m = n$ and consider the coordinate vectors $\vec{e}_1 = \underset{(n\times 1)}{\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}}$, $\vec{e}_2 = \begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix}$, $\dots$, $\vec{e}_n = \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}$.

Then $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ are <u>linearly independent</u>.

(ii) Given any $\vec{v}_1, \dots, \vec{v}_n$ in $\mathbb{R}^m$ such that one of them, say $\vec{v}_1$, is $\vec{0}$; the collection $\vec{v}_1 = \vec{0}, \vec{v}_2, \dots, \vec{v}_n$ is <u>linearly dependent</u>. This is true because $x_1 = 1, x_2 = 0, x_3 = 0, \dots, x_n = 0$ solves $x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_n\vec{v}_n = \vec{0}$.

(iii) Let $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix}3\\4\end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix}-1\\2\end{bmatrix}$ in $\mathbb{R}^2$.

Then $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are <u>linearly dependent</u>. This is because if <u>#vectors > dimension</u>, then they are <u>always dependent</u>.

To find a linear combination $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$, we write the corresponding homogeneous system:

$$\underbrace{\boxed{\begin{matrix}1\\2\end{matrix}}}_{\vec{v}_1}\begin{matrix}x_1\\x_1\end{matrix} + \underbrace{\boxed{\begin{matrix}3\\4\end{matrix}}}_{\vec{v}_2}\begin{matrix}x_2\\x_2\end{matrix} + \underbrace{\boxed{\begin{matrix}(-1)\\2\end{matrix}}}_{\vec{v}_3}\begin{matrix}x_3 = 0\\x_3 = 0\end{matrix}$$

and bring the coefficient matrix to its reduced echelon form.

$$\begin{bmatrix}1 & 3 & -1\\2 & 4 & 2\end{bmatrix} \xrightarrow[R_2 \to R_2 - 2R_1]{} \begin{bmatrix}1 & 3 & -1\\0 & -2 & 4\end{bmatrix} \xrightarrow[R_2 \to -\frac{1}{2}R_2]{} \begin{bmatrix}1 & 3 & -1\\0 & 1 & -2\end{bmatrix} \xrightarrow[R_1 \to R_1 - 3R_2]{} \begin{bmatrix}1 & 0 & 5\\0 & 1 & -2\end{bmatrix}$$

Thus the given system becomes

$$x_1 + 5x_3 = 0$$
$$x_2 - 2x_3 = 0.$$

So $\boxed{\begin{array}{l} x_1 = -5t \\ x_2 = 2t \\ x_3 = t \end{array}}$ solves it and we find a non-trivial solution:

$t \in \mathbb{R}$ any.

$x_1 = -5,\ x_2 = 2,\ x_3 = 1$. That is,

$$\boxed{-5\begin{bmatrix}1\\2\end{bmatrix} + 2\begin{bmatrix}3\\4\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}}$$

Hence, for example, $\vec{v}_3$ can be written as a linear combination of $\vec{v}_1$ and $\vec{v}_2$, as:

$$\boxed{\vec{v}_3 = 5\vec{v}_1 - 2\vec{v}_2}.$$

(9.2) We also defined singular and non-singular matrices.

A square matrix $\underset{(n\times n)}{A = (a_{ij})}$ is called non-singular if $A\vec{x} = \vec{0}$ has only trivial solution. (singular if non-trivial solutions can be found). We proved

$$\boxed{\begin{array}{ll} \text{Non-singular} & \leftrightarrow \text{Columns are linearly independent} \\ & \leftrightarrow \text{Invertible} \end{array}}$$

— END of material for Mid Term 1 —