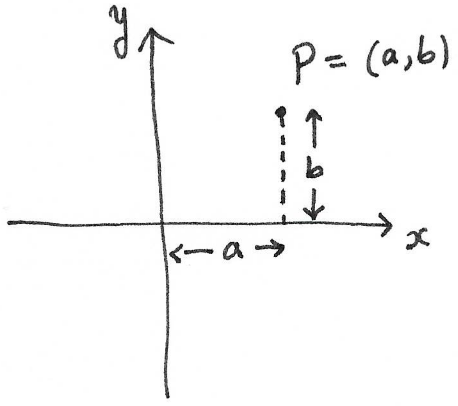
Lecture 9 continued

(9.3) Now we will focus solely on vectors in 2 and 3 dimensions, as you may have encountered in a previous Calculus course. The idea is that in dimensions 2 & 3 vectors can be drawn and the algebraic operations have clear geometric meaning. Later in the course we are going to have to let go of this geometric intuition (dim ≥ 4) where algebraic manipulations are the only things at our disposal.

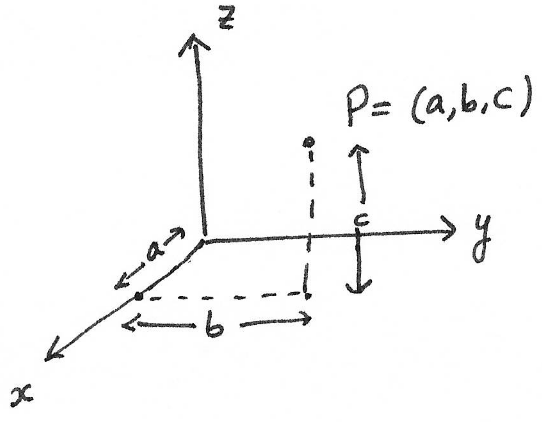
Historically speaking the idea of vectors comes from physics ($\sim$ 1690's) after Newton's laws of motion were written down. The first "vector" was force acting on a particle — where in order to accurately apply Newton's principles - one needs to know not only the "magnitude" but also "the direction" in which the force is acting

(9.4) Coordinate systems in 2 and 3 dimensions.

A point in $\mathbb{R}^2$ (in $\mathbb{R}^3$) is specified by 2 real numbers (3 real numbers).

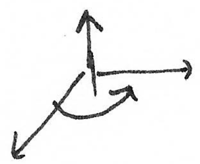


$\mathbb{R}^2$



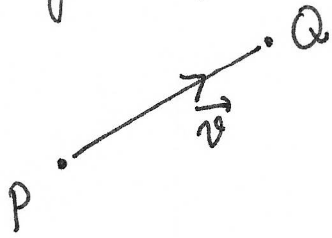
$\mathbb{R}^3$

Recall the <u>right hand rule</u> for (x, y, z) coordinate system in $\mathbb{R}^3$.
(if you place your right hand along x-axis in such a way that your fingers can move towards y-axis, then your thumb will point in the z-direction:



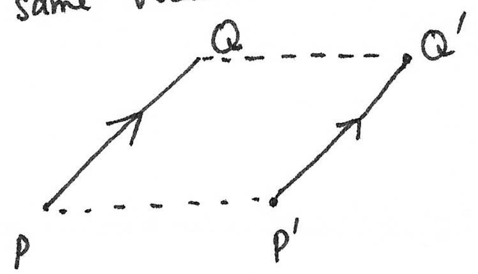
)

(9.5) Geometrically a vector is represented as a <u>line segment</u> (directed) joining two points P and Q.



Starting point (or tail of $\vec{v}$) $= P$
Terminal point (or head of $\vec{v}$) $= Q$.
Sometimes written as $\vec{v} = \overrightarrow{PQ}$.

Observations. – (i) We <u>don't</u> distinguish between parallel line segments, pointing in the same direction, of same length – as they give the same vectors.

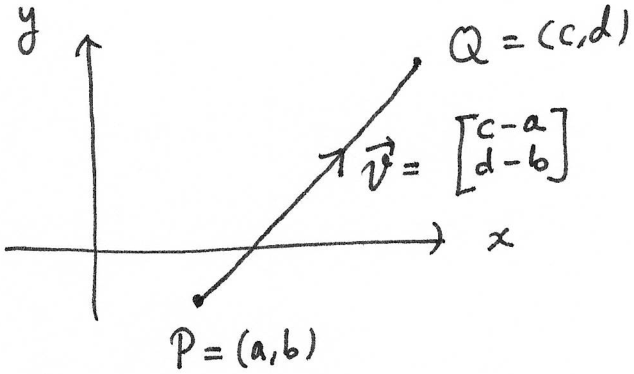


: $\overrightarrow{PQ} = \overrightarrow{P'Q'}$

(ii) $\vec{0}$ = the only vector of magnitude 0 and no direction
$= \overrightarrow{PP}$ for any point P.

As a 2x1 matrix, a vector represented geometrically by joining $P = (a, b)$ to $Q = (c, d)$ in (x,y)-plane is given by

$$\overrightarrow{PQ} = \begin{bmatrix} c-a \\ d-b \end{bmatrix}$$



Similary in 3 dimensions, if $P = (a_1, b_1, c_1)$ then
$Q = (a_2, b_2, c_2)$

$$\overrightarrow{PQ} = \begin{bmatrix} a_2-a_1 \\ b_2-b_1 \\ c_2-c_1 \end{bmatrix}$$ "difference of coordinates".

e.g. if $P = (1,1,1)$ and $\overrightarrow{PQ} = \vec{v} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$, then
$Q = (1+2,\ 1+(-1),\ 1+4) = (3,0,5)$.

Magnitude, or length, or norm: If $\vec{v} = \overrightarrow{PQ}$, then $\|\vec{v}\|$ is the distance from P to Q. That is,

$$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix} \rightsquigarrow \|\vec{v}\| = \sqrt{a^2+b^2} \quad ; \quad \vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightsquigarrow \|\vec{v}\| = \sqrt{a^2+b^2+c^2}$$

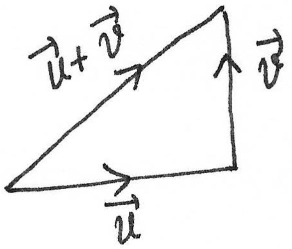
2D 3D

(9.6) Addition and scalar multiplication. –

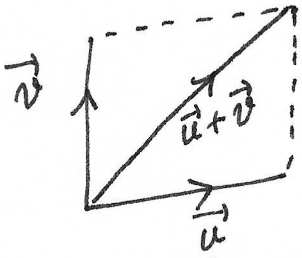
Algebraically – vectors viewed as 2×1 or 3×1 matrices are scaled and added componentwise : EASY !

$$5\cdot\begin{bmatrix}2\\1\\-1\end{bmatrix}+3\cdot\begin{bmatrix}-2\\-4\\7\end{bmatrix}=\begin{bmatrix}5(2)+3(-2)\\5(1)+3(-4)\\5(-1)+3(7)\end{bmatrix}=\begin{bmatrix}4\\-7\\16\end{bmatrix}.$$

Geometrically : we have "triangle & parallelogram laws" of vector addition :



(triangle law)

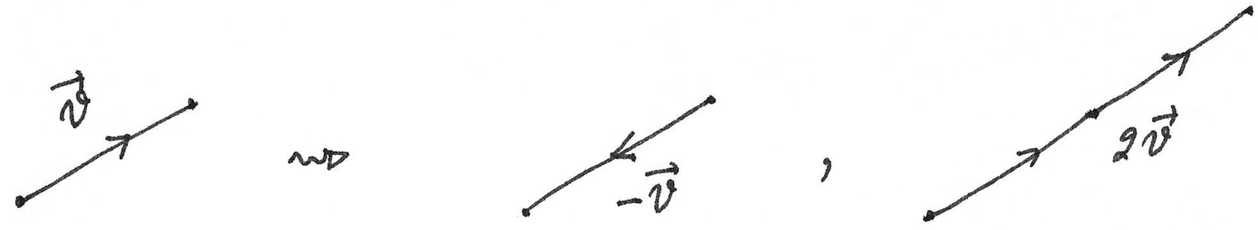


(parallelogram law)

Triangle law : By moving $\vec{v}$ parallel to itself, make sure tail of $\vec{v}$ = head of $\vec{u}$. $\vec{u}+\vec{v}$ then starts at tail of $\vec{u}$ and ends at head of $\vec{v}$.

Parallelogram law : If $\vec{u}$ & $\vec{v}$ start at the same point, complete the parallelogram, $\vec{u}+\vec{v}$ starts at the common tail of $\vec{u}$ & $\vec{v}$ ends at the opposite point in the parallelogram

Scaling a vector, geometrically, changes the length of the segment while retaining the same direction (if scalar >0). flips the direction (if scalar <0).



(Scalar multiplication - geometrically).

(9.7) <u>Coordinate vectors</u> (or basic unit vectors).

$\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$ in $\mathbb{R}^2$

$\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ $\vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$ in $\mathbb{R}^3$.

(In Calculus, we use $\vec{i}$, $\vec{j}$, $\vec{k}$ for these; and write vectors as pairs/triples of numbers sorrounded by angular brackets

$\begin{bmatrix}1\\-1\\4\end{bmatrix}$ in this course $\longleftrightarrow$ $\langle 1, -1, 4 \rangle$ in Calculus.)

Unit vector = any vector of length 1.

e.g. find <u>the</u> unit vector in the same direction as $\begin{bmatrix}1\\4\\7\end{bmatrix}$.

$$\vec{v} = \begin{bmatrix}1\\4\\7\end{bmatrix} \rightsquigarrow \|\vec{v}\| = \sqrt{1^2+4^2+7^2} = \sqrt{1+16+49}$$
$$= \sqrt{66}.$$

Answer: $\dfrac{1}{\|\vec{v}\|} \cdot \vec{v} = \begin{bmatrix}1/\sqrt{66}\\4/\sqrt{66}\\7/\sqrt{66}\end{bmatrix}$.