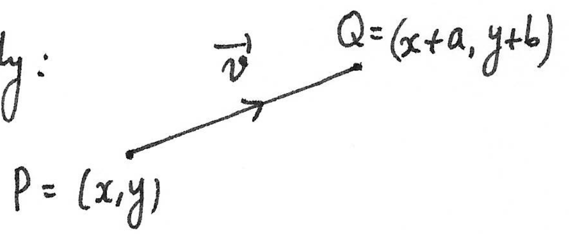
Lecture 10

(10.0) Recall: last time we discussed vectors in 2 & 3 dimensions geometrically.

Algebraically: $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ $(\mathbb{R}^2)$, $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ $(\mathbb{R}^3)$

Geometrically:



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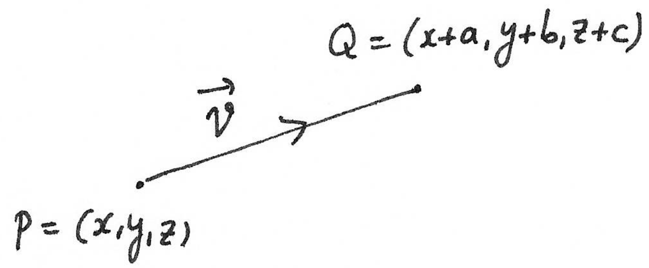
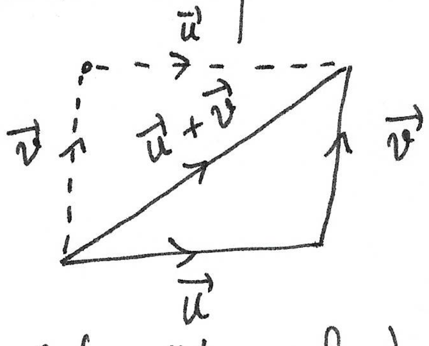


We discussed: magnitude, or length, or norm of a vector

$$\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}, \quad \|\vec{v}\| = \sqrt{a^2+b^2} \quad \text{in } \mathbb{R}^2.$$

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \|\vec{v}\| = \sqrt{a^2+b^2+c^2} \quad \text{in } \mathbb{R}^3.$$

- geometric interpretation of addition and scalar multiplication:



(triangle/parallelogram law)

$$\|c\vec{v}\| = |c| \cdot \|\vec{v}\| \quad (c \in \mathbb{R})$$

$$\text{Direction of } c\vec{v} = \begin{cases} \text{same as } \vec{v} & \text{if } c>0 \\ \text{opposite to } \vec{v} & \text{if } c<0 \\ \text{unspecified} & \text{if } c=0 \end{cases}$$

- Basic unit vectors: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$.

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ in } \mathbb{R}^3.$$

- Given a non-zero vector $\vec{v}$, there is a unique <u>unit vector</u> in the direction same as $\vec{v}$, namely $\frac{1}{\|\vec{v}\|} \cdot \vec{v}$.

(10.1) Today we are going to talk about <u>dot product</u>.

In $\mathbb{R}^2$: $\vec{u} = \langle a_1, b_1 \rangle$ (Calculus notation) $= \begin{bmatrix} a_1 \\ b_1 \end{bmatrix}$ (matrix notation), $\vec{v} = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$

$$\boxed{\vec{u} \cdot \vec{v} = a_1 a_2 + b_1 b_2}$$

In $\mathbb{R}^3$: $\vec{u} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$

$$\boxed{\vec{u} \cdot \vec{v} = a_1 a_2 + b_1 b_2 + c_1 c_2}$$

Note - We saw that this formula generalizes to any dimension, and can be written as matrix multiplication:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \rightsquigarrow \boxed{\vec{u}^T \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 + \ldots + u_n v_n}$$

For now, we will stick to Calculus notation $\vec{u} \cdot \vec{v}$, but later in the course we will solely use $\vec{u}^T \vec{v}$ for this quantity.

(10.2) Properties of the dot product.

(i) $\|\vec{u}\|^2 = \vec{u} \cdot \vec{u}$

(ii) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

(iii) $\vec{u} \cdot (\vec{v}_1 + \vec{v}_2) = \vec{u} \cdot \vec{v}_1 + \vec{u} \cdot \vec{v}_2$

For any vectors $\vec{u}, \vec{v}, \vec{v}_1, \vec{v}_2$

These follow directly from the definition.

Example. - $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v}) \cdot (\vec{u}+\vec{v})$ (using (i))

$= \vec{u} \cdot \vec{u} + \vec{u} \cdot \vec{v} + \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v}$ (using (iii))

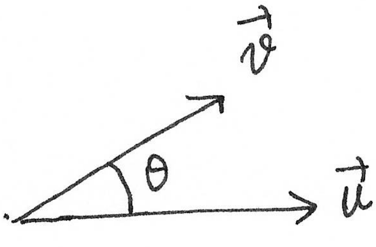
$= \|\vec{u}\|^2 + 2\,\vec{u} \cdot \vec{v} + \|\vec{v}\|^2$ (using (ii) & (i))

$$\boxed{\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\,\vec{u} \cdot \vec{v}}$$

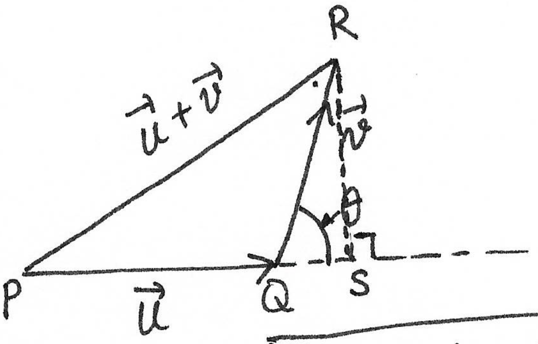
(10.3) Geometric formula for the dot product.

If θ = angle between $\vec{u}$ and $\vec{v}$ $(0 \leq \theta \leq \pi)$

Then $\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \; \|\vec{v}\| \cos(\theta)}$



<u>Proof</u> We are going to compute $\|\vec{u}+\vec{v}\|^2$ geometrically and compare it with the formula written on the last page



$$\boxed{\begin{aligned}|RS| &= |RQ|\sin\theta \\ |QS| &= |RQ|\cos\theta\end{aligned}}$$

$$\begin{aligned}
\|\vec{u}+\vec{v}\|^2 &= |PR|^2 \\
&= |PS|^2 + |RS|^2 \\
&= (\|\vec{u}\| + \|\vec{v}\|\cos\theta)^2 \\
&\quad + (\|\vec{v}\|\sin\theta)^2 \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2(\cos^2\theta + \sin^2\theta) \\
&\quad + 2\|\vec{u}\|\,\|\vec{v}\|\cos(\theta) \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\|\vec{u}\|\,\|\vec{v}\|\cos(\theta) \\
&= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 2\,\vec{u}\cdot\vec{v} \quad \text{(from last page)}
\end{aligned}$$

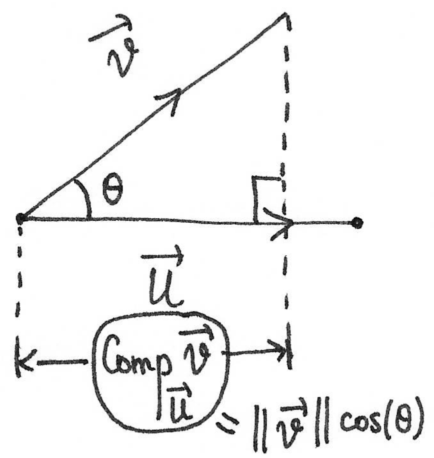
Hence $\boxed{\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos(\theta)}$ □

(10.4) Orthogonal ~~projection~~ vectors and projection.

We say two vectors $\vec{u}$ and $\vec{v}$ are <u>orthogonal</u> (or perpendicular), denoted by $\vec{u}\perp\vec{v}$, if

$$\boxed{\vec{u}\cdot\vec{v} = 0}$$

In physical problems, we sometimes have to compute "component of $\vec{v}$ along $\vec{u}$", or "projection of $\vec{v}$ onto $\vec{u}$"



Component of $\vec{v}$ along $\vec{u}$

$$= \text{Comp}_{\vec{u}}\vec{v} = \|\vec{v}\| \cos(\theta)$$

$$= \cancel{\|\vec{v}\|} \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \cancel{\|\vec{v}\|}} \right)$$

$$\boxed{\text{Comp}_{\vec{u}}\vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|}}$$

Projection of $\vec{v}$ onto $\vec{u}$ $= \begin{pmatrix} \text{Component of } \vec{v} \\ \text{along } \vec{u} \end{pmatrix} \cdot \begin{pmatrix} \text{Unit vector} \\ \text{along } \vec{u} \end{pmatrix}$

$$\text{Proj}_{\vec{u}}\vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|} \left(\frac{1}{\|\vec{u}\|} \vec{u} \right)$$

$$\boxed{\text{Proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \right) \vec{u}}$$

(10.5) Examples. (i) Show that $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \vec{v}$ are orthogonal to each other. (Soln: $\vec{u} \cdot \vec{v} = 1(-2) + 0(3) + 2(1) = 0$.
$\Rightarrow \vec{u} \perp \vec{v}$.)

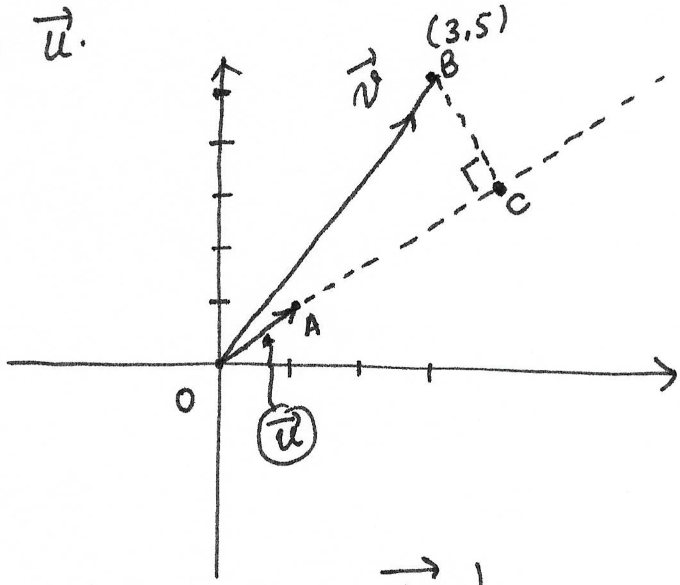
(ii) Find the angle between $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ & $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|} = \frac{1(2) + 1(-1) + 0(2)}{\sqrt{1^2+1^2+0^2} \, \sqrt{2^2+(-1)^2+2^2}} = \frac{1}{3\sqrt{2}}.$$

$$\theta = \cos^{-1}\left(\frac{1}{3\sqrt{2}}\right) \left(= 76.37^\circ \text{ or } 1.33 \text{ radians}\right).$$

(iii) Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Compute the projection of $\vec{v}$ onto $\vec{u}$.



$$\text{Proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2}\right)\vec{u}$$

$$= \frac{1(3) + 1(5)}{1^2 + 1^2} \, \vec{u}$$

$$= \frac{8}{2} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

$\left(\text{Proj}_{\vec{u}}(\vec{v}) = \overrightarrow{OC}\right)$