Lecture 11

(11.0) Recall – we have been reviewing vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$. In the last lecture we studied <u>the dot product</u>

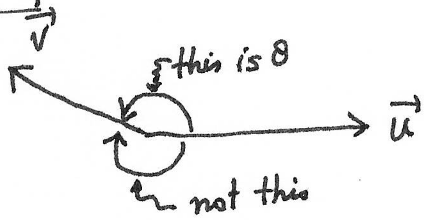
$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \rightsquigarrow \vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 \in \mathbb{R}.$$

(i) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (ii) $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

(iii) $\vec{u} \cdot (\vec{v}_1 + \vec{v}_2) = \vec{u} \cdot \vec{v}_1 + \vec{u} \cdot \vec{v}_2$

<u>Cosine formula</u> : $\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \cdot \|\vec{v}\| \cdot \cos(\theta)}$

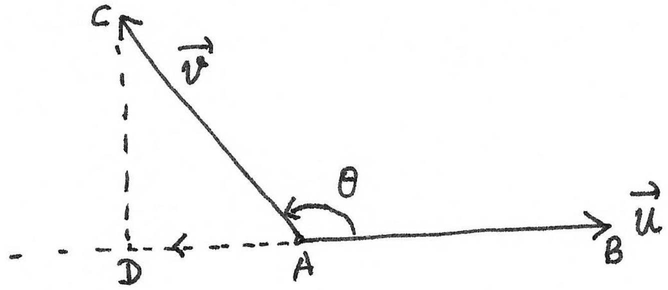
where θ = angle between $\vec{u}$ and $\vec{v}$ $(0 \leq \theta \leq \pi)$



(iv) $\vec{u} \perp \vec{v} \longleftrightarrow \vec{u} \cdot \vec{v} = 0$.

(read: $\vec{u}$ and $\vec{v}$ are <u>orthogonal</u> to each other)

(v) <u>Projection</u> of <u>$\vec{v}$ onto $\vec{u}$</u>



$$\boxed{\mathrm{Proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right) \vec{u}}$$

$(\mathrm{Proj}_{\vec{u}}(\vec{v}) = \overrightarrow{AD})$

$$\vec{v} = \overrightarrow{AC} = \underbrace{\overrightarrow{AD}}_{\mathrm{Proj}_{\vec{u}}(\vec{v})} + \underbrace{\overrightarrow{DC}}_{\text{Component of } \vec{v} \text{ orthogonal to } \vec{u}}$$

(denoted by $\mathrm{Orth}_{\vec{u}}(\vec{v})$)

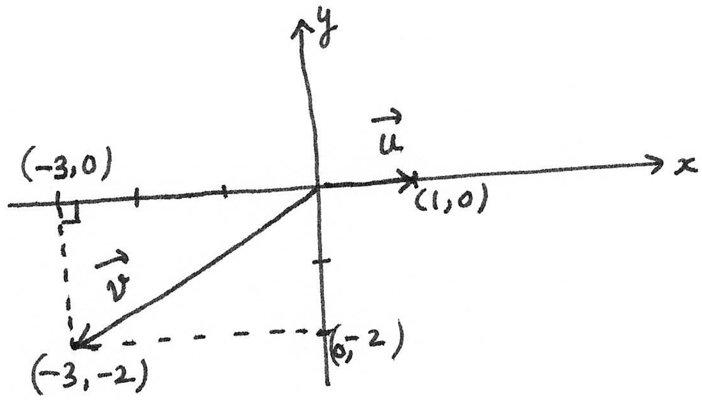
$$\boxed{\begin{aligned} \mathrm{Orth}_{\vec{u}}(\vec{v}) &= \vec{v} - \mathrm{Proj}_{\vec{u}}(\vec{v}) \\ &= \vec{v} - \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right) \vec{u} \end{aligned}}$$

(11.1) Example. – Let $\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$

$$\text{Proj}_{\vec{u}}(\vec{v}) = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$\text{Orth}_{\vec{u}}(\vec{v}) = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

as easily seen in the picture



Example. Let $\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ (Picture here will be a bit obscure, and not help with computation).

$$\text{Proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$$

$$= \left(\frac{2+3+4}{1+1+1}\right)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \quad (\text{parallel to } \vec{u} \checkmark)$$

$$\text{Orth}_{\vec{u}}(\vec{v}) = \vec{v} - \text{Proj}_{\vec{u}}(\vec{v}) = \begin{bmatrix} 2-3 \\ 3-3 \\ 4-3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\left(\vec{u} \perp \text{Orth}_{\vec{u}}(\vec{v}) \text{ because } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -1+0+1 = 0 \checkmark\right)$

This is true in general anyway by the way we defined $\text{Orth}_{\vec{u}}(\vec{v})$

$$\vec{u}\cdot\left(\text{Orth}_{\vec{u}}(\vec{v})\right) = \vec{u}\cdot\left(\vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}\right)$$

$$= \vec{u}\cdot\vec{v} - \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)(\cancel{\vec{u}\cdot\vec{u}}) = 0 \checkmark.$$

(11.2) Cross product.

<u>Only defined for vectors in $\mathbb{R}^3$</u>

$\vec{u}, \vec{v}$ in $\mathbb{R}^3 \rightsquigarrow \vec{u} \times \vec{v} \in \mathbb{R}^3$ ($\perp$ to both $\vec{u}$ & $\vec{v}$)

<u>Definition</u> If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, then

$$\boxed{\vec{u} \times \vec{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}}.$$

This formula is usually remembered in terms of a <u>determinant</u>. We will discuss determinant of an arbitrary $n \times n$ size matrix later in the course. But for $n = 2$ and 3, we can see it right now.

$$\boxed{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc}$$

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = a_1 \cdot \det \underbrace{\begin{bmatrix} b_2 & b_3 \\ c_2 & c_3 \end{bmatrix}}_{\text{2x2 matrix obtained by removing 1st row and 1st column}} - a_2 \det \overbrace{\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix}}^{\text{2x2 matrix after removing 1st row \& 2nd column}} + a_3 \det \underbrace{\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}}_{\text{2x2 matrix after removing 1st row and 3rd column}}$$

For example -
$$\det \begin{bmatrix} 1 & 0 & -1 \\ 3 & 2 & 4 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= 1 \cdot \det \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix} - 0 \cdot \det \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} + (-1) \cdot \det \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= 1\,(2-4) + (-1)(3-2) = -2-1 = -3.$$

$$\vec{u} \times \vec{v} = \det \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \qquad \left(\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \right)$$

$$= \underbrace{(u_2 v_3 - u_3 v_2)}_{\det \begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix}} \vec{e}_1 - \underbrace{(u_1 v_3 - u_3 v_1)}_{\det \begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix}} \vec{e}_2 + \underbrace{(u_1 v_2 - u_2 v_1)}_{\det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix}} \vec{e}_3$$

(11.3) Properties of cross product. (follow easily from the formula).

(i) $\vec{u} \times \vec{u} = \vec{0}$

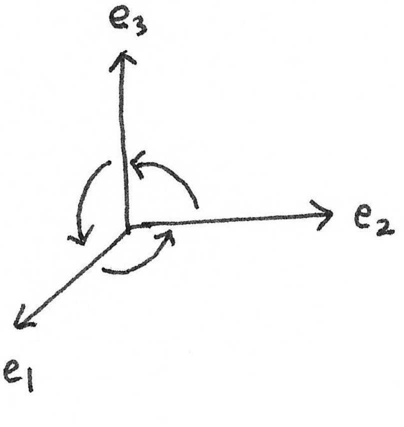
(ii) $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

(iii) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

(iv) $\vec{u} \times (\lambda \vec{v}) = \lambda (\vec{u} \times \vec{v})$

here $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$

Easy calculation :
$$\boxed{\begin{aligned} \vec{e}_1 \times \vec{e}_2 &= \vec{e}_3 \\ \vec{e}_2 \times \vec{e}_3 &= \vec{e}_1 \\ \vec{e}_3 \times \vec{e}_1 &= \vec{e}_2 \end{aligned}}$$



Hence, cross product of coordinate vectors $\vec{e}_1$, $\vec{e}_2$ & $\vec{e}_3$ can be easily remembered as the right hand rule.

Sometimes people prefer to remember this, and properties (i), (ii), (iii) & (iv), which can be used to compute the cross product.

Example. $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \vec{e}_1 - \vec{e}_2 + 3\vec{e}_3$

$\vec{v} = \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix} = 4\vec{e}_1 + 7\vec{e}_2$

$$\vec{u} \times \vec{v} = (\vec{e}_1 - \vec{e}_2 + 3\vec{e}_3) \times (4\vec{e}_1 + 7\vec{e}_2)$$

$$= \boxed{\vec{e}_1 \times (4\vec{e}_1)} + \vec{e}_1 \times (7\vec{e}_2) - \vec{e}_2 \times (4\vec{e}_1) - \boxed{\vec{e}_2 \times (7\vec{e}_2)}$$
$$+ (3\vec{e}_3) \times (4\vec{e}_1) + (3\vec{e}_3) \times (7\vec{e}_2)$$

(vectors in $\square$ are $\vec{0}$ by (i) and (iv))

$$= 7(\vec{e}_1 \times \vec{e}_2) - 4(\vec{e}_2 \times \vec{e}_1)$$
$$+ 12(\vec{e}_3 \times \vec{e}_1) + 21(\vec{e}_3 \times \vec{e}_2)$$

$$= 7\vec{e}_3 + 4\vec{e}_3 + 12\vec{e}_2 - 21\vec{e}_1$$

$$\left(\begin{aligned} e_1 \times e_2 &= e_3 \\ e_2 \times e_1 &= -e_1 \times e_2 \\ &= -e_3 \\ &\dots \end{aligned} \right)$$

$$= -21\vec{e}_1 + 12\vec{e}_2 + 11\vec{e}_3$$

$$= \begin{bmatrix} -21 \\ 12 \\ 11 \end{bmatrix}.$$

(11.4) $\underline{\vec{u} \times \vec{v} \text{ is orthogonal to both } \vec{u} \text{ \& } \vec{v}}$.

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

$$= u_1 (u_2 v_3 - u_3 v_2) + u_2 (u_3 v_1 - u_1 v_3) + u_3 (u_1 v_2 - u_2 v_1)$$

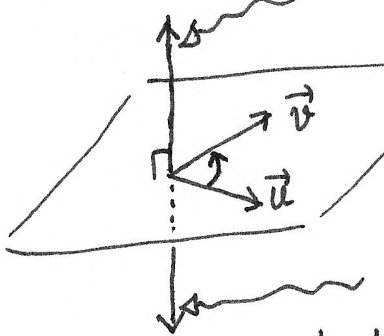
$$= v_1 (u_2 u_3 - u_2 u_3) + v_2 (-u_1 u_3 + u_1 u_3) + v_3 (u_1 u_2 - u_1 u_2) = 0 \checkmark$$

Switch the roles of $\vec{u}$ and $\vec{v}$ to see that $\vec{v} \cdot (\vec{u} \times \vec{v})$

$$= -\vec{v} \cdot (\vec{v} \times \vec{u}) = 0.$$

Thus, given $\vec{u}$ and $\vec{v}$, the direction of $\vec{u} \times \vec{v}$ is $\perp$ to the plane in which $\vec{u}$ and $\vec{v}$ lie - given by the right hand rule

$\vec{u} \times \vec{v}$ points this way - by the right hand rule



NOT the direction of $\vec{u} \times \vec{v}$, but that of $\vec{v} \times \vec{u}$.

(Two directions $\perp$ to the plane containing $\vec{u}$ and $\vec{v}$)

Now that we know the direction of $\vec{u} \times \vec{v}$, we will have to figure out its magnitude - to understand it geometrically.

(11.5) Sine formula.

$$\boxed{\|\vec{u}\times\vec{v}\| = \|\vec{u}\|\,\|\vec{v}\|\,\sin(\theta)}$$

($0 \le \theta \le \pi$ is the angle between $\vec{u}$ & $\vec{v}$).

The proof of this formula is very algebraic. Namely, we check that

$$\|\vec{u}\times\vec{v}\|^2 + (\vec{u}\cdot\vec{v})^2 = \|\vec{u}\|^2 \cdot \|\vec{v}\|^2 \quad - \quad (*)$$

and use the cosine formula for the dot product. (& $\sin^2\theta + \cos^2\theta = 1$).

$$\text{L.H.S. of } (*) = (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 + (u_1v_1 + u_2v_2 + u_3v_3)^2$$

$$= u_2^2v_3^2 + u_3^2v_2^2 - \cancel{2u_2u_3v_2v_3} + u_3^2v_1^2 + u_1^2v_3^2 - \cancel{2u_1u_3v_1v_3} + u_1^2v_2^2 + u_2^2v_1^2 - \cancel{2u_1u_2v_1v_2} + u_1^2v_1^2 + u_2^2v_2^2 + u_3^2v_3^2$$

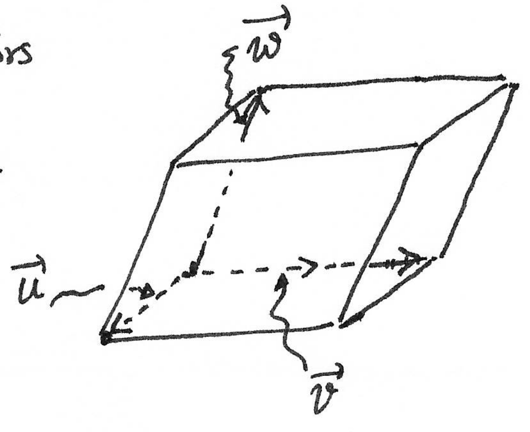
$$+ \cancel{2u_1u_2v_1v_2} + \cancel{2u_1u_3v_1v_3} + \cancel{2u_2u_3v_2v_3}$$

$$= u_1^2(v_1^2+v_2^2+v_3^2) + u_2^2(v_1^2+v_2^2+v_3^2) + u_3^2(v_1^2+v_2^2+v_3^2)$$

$$= (u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) = \|\vec{u}\|^2 \cdot \|\vec{v}\|^2 = \text{R.H.S. of } (*).$$

(11.6) Applications. - (i) We can test whether two vectors lie on a line (collinear) by checking if their cross product is $\vec{0}$. Although collinearity of two vectors can be more easily checked by observing if one is a scalar multiple of other or not.

(ii) Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be three vectors in $\mathbb{R}^3$. Let P be the parallelopipe formed by using these vectors as sides



Then $\boxed{\text{Volume}(P) = |\vec{u} \cdot (\vec{v} \times \vec{w})|}$

Example.- Determine whether $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 7 \\ 3 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 5 \\ 7 \\ 1 \end{bmatrix}$

are coplanar (i.e. lie on the same plane) or not.

<u>Sol</u> Three vectors are coplanar if and only if the volume of the parallelopipe formed by using them as sides is 0.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 3 & 7 & 3 \\ 5 & 7 & 1 \end{vmatrix} = (7-21)\vec{e}_1 - (3-15)\vec{e}_2 + (21-35)\vec{e}_3$$

$$= \begin{bmatrix} -14 \\ 12 \\ -14 \end{bmatrix}$$

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = 1(-14) + 0(12) + (-1)(-14) = 0.$$

So, $\vec{u}$, $\vec{v}$ and $\vec{w}$ are coplanar.

<u>Remarks</u> (i) Note: $\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$

$$= (\vec{u} \times \vec{v}) \cdot \vec{w}$$

(ii) $(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$ (See

e.g. $\vec{u} = \vec{e}_1$, $\vec{v} = \vec{e}_1$, $\vec{w} = \vec{e}_2$. L.H.S. $= \vec{0}$ since $\vec{e}_1 \times \vec{e}_1 = 0$

R.H.S. $= \vec{e}_1 \times (\vec{e}_1 \times \vec{e}_2) = \vec{e}_1 \times \vec{e}_3 = -\vec{e}_2$.