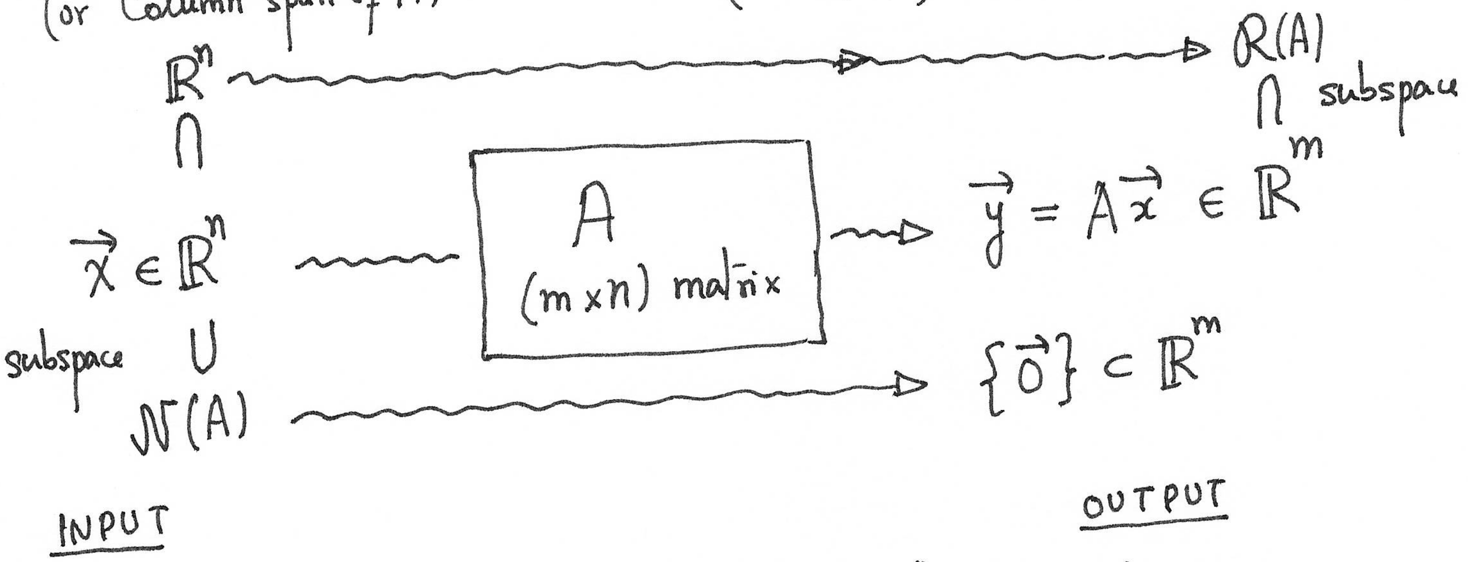
(15.0) Recall – last time we defined two important types of subspaces coming from an $m \times n$ matrix A.

Null space of A : $\mathcal{N}(A) = \{ \vec{x} \text{ in } \mathbb{R}^n : A\vec{x} = \vec{0} \} \subset \mathbb{R}^n$

Range of A (or Column span of A) : $\mathcal{R}(A) = \{ \vec{y} \text{ in } \mathbb{R}^m : \vec{y} = A\vec{x} \text{ for some } \vec{x} \text{ in } \mathbb{R}^n \}$ $(\subset \mathbb{R}^m)$



We discussed some examples to "describe" these subspaces.

- Algebraic description of a subspace $V \subset \mathbb{R}^N$ involves giving conditions on entries of a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{bmatrix} \in \mathbb{R}^N$ for it to be in V.
- Another way (preferred for some applications – as we will see later) is to give a basis of V.

(15.1) Basis of a subspace $V \subset \mathbb{R}^N$ (N: positive integer)

A subset $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ of V is said to be a basis of V if:

(1) Every vector $\vec{w}$ in V can be written as a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p$. That is, we can find numbers $a_1, a_2, \dots, a_p \in \mathbb{R}$ such that

$$\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_p\vec{v}_p.$$

We say $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ spans V

(2) The set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ is linearly independent. Recall that this meant the following:

if $x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_p\vec{v}_p = \vec{0}$, then $x_1 = x_2 = \dots = x_p = 0$.

A subset of V satisfying condition (1) alone (and not necessarily (2)) is called a spanning set of vectors.

Examples. – (i) $V = \mathbb{R}^2$. $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis. Let us check this.

(Spanning Condition). If $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, then $\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. ✓

(Linear independence). If $x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ then

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $x_1 = x_2 = 0$. ✓

(ii) Similarly $\left\{ \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e}_N = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$
$(N \times 1)$

is a basis of $\mathbb{R}^N$.

(iii) $V = \mathbb{R}^2$ again $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is also a basis of $\mathbb{R}^2$. (Basis of a vector space is <u>NOT</u> unique).

(Spanning) Given $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in \mathbb{R}^2$, let us find x and y so that

$$x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}. \text{ That is } \begin{aligned} x + y &= b_1 \\ x - y &= b_2 \end{aligned}$$

This system is easily solved as $x = \dfrac{b_1 + b_2}{2}$, $y = \dfrac{b_1 - b_2}{2}$.

(linear independence) If $x \begin{bmatrix} 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, by the calculation done above with $b_1 = b_2 = 0$, we get $x = y = 0$.

(iv) $V = \mathbb{R}^3$. $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \subset V$.

B cannot possibly be a basis, since it has 4 vectors (in $\mathbb{R}^3$) so they will have to be linearly dependent.

Let us find a linear relation among these 4 vectors.

$$\vec{V}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{V}_2 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \quad \vec{V}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \vec{V}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$x_1\vec{v_1} + x_2\vec{v_2} + x_3\vec{v_3} + x_4\vec{v_4} = \vec{0}$ is same as the following homogeneous system.

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 3 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3 \to R_3 - R_1]{\text{Omitting the Last column}} \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 \to R_3 - R_2]{} \underbrace{\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{echelon form}} \xrightarrow[R_1 \to R_1 - 2R_2]{} \underbrace{\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{reduced echelon form.}}$$

i.e.

$$x_1 + 3x_3 = 0$$
$$x_2 - x_3 = 0$$
$$x_4 = 0$$

So, general solution of this homogeneous system is $x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

We obtain (set $x_3 = 1$) : $-3\vec{v_1} + \vec{v_2} + \vec{v_3} = \vec{0}$.

Notice : this means, say $\vec{v_2} = 3\vec{v_1} - \vec{v_3}$ is "redundant".

That is $\{\vec{v_1}, \vec{v_3}, \vec{v_4}\}$ span the same subspace as $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$.

Check: $\{\vec{v_1}, \vec{v_3}, \vec{v_4}\}$ is linearly independent.

(15.2) Some properties of a basis of a subspace V.

Let $B = \{\vec{V_1}, \vec{V_2}, \ldots, \vec{V_p}\}$ be a basis of V (subspace of $\mathbb{R}^N$).

<u>Properties</u>. (1) (Uniqueness of expression). For any $\vec{w}$ in V, there are <u>uniquely determined</u> numbers $a_1, a_2, \ldots, a_p \in \mathbb{R}$ such that $\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p$.

(Proof: If we have two such expressions,

$$\vec{w} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_p\vec{v}_p$$

then taking their difference, we get

$$(a_1 - b_1)\vec{v}_1 + (a_2 - b_2)\vec{v}_2 + \cdots + (a_p - b_p)\vec{v}_p = \vec{0}$$

Since these vectors are linearly independent, we get

$$a_1 - b_1 = a_2 - b_2 = \cdots = a_p - b_p = 0$$

$$a_1 = b_1 \text{ and } a_2 = b_2 \text{ and } \cdots \; a_p = b_p. \quad \square)$$

(2) Let $S = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q\} \subset V$ be another set of q-vectors with $q > p$. Then S is linearly dependent.

(compare - a collection of m vectors in $\mathbb{R}^n$, with $m > n$, is always linearly dependent.)

(Proof. Write each $\vec{w}$ as a unique linear combination of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$:

$$\begin{aligned}
\vec{w}_1 &= a_{11}\vec{v}_1 + a_{12}\vec{v}_2 + \ldots + a_{1p}\vec{v}_p \\
\vec{w}_2 &= a_{21}\vec{v}_1 + a_{22}\vec{v}_2 + \ldots + a_{2p}\vec{v}_p \\
&\vdots \\
\vec{w}_q &= a_{q1}\vec{v}_1 + a_{q2}\vec{v}_2 + \ldots + a_{qp}\vec{v}_p
\end{aligned}
\rightsquigarrow A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ & & \vdots & \\ a_{q1} & a_{q2} & \cdots & a_{qp} \end{bmatrix}$$

$(q \times p)$-size matrix

Consider $x_1\vec{w}_1 + x_2\vec{w}_2 + \ldots + x_q\vec{w}_q = \vec{0}$. Substituting expressions of $\vec{w}$'s in terms of $\vec{v}$'s gives us a homogeneous system as follows:

$$\begin{aligned}
&(a_{11}x_1 + a_{21}x_2 + \ldots + a_{q1}x_q)\vec{v}_1 \\
+ &(a_{12}x_1 + a_{22}x_2 + \ldots + a_{q2}x_q)\vec{v}_2 \\
+ &\ldots + (a_{1p}x_1 + a_{2p}x_2 + \ldots + a_{qp}x_q)\vec{v}_p = \vec{0}.
\end{aligned}$$

As $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ are linearly independent, we must have

$$\begin{aligned}
a_{11}x_1 + a_{21}x_2 + \ldots + a_{q1}x_q &= 0 \\
&\vdots \\
a_{1p}x_1 + a_{2p}x_2 + \ldots + a_{qp}x_q &= 0
\end{aligned}
\qquad \text{i.e. } \boxed{A^T \cdot \vec{x} = \vec{0}}$$

Since $p < q$, we have less equations and more unknowns.
Hence, there must be infinitely many solutions.
So $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q\}$ has to be a linearly dependent set.)

(15.3) Dimension of a subspace $V \subset \mathbb{R}^N$.

Theorem. If $B_1 = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ and

$B_2 = \{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_q\}$ are two bases

plural of basis is bases

of V, then $p = q$.

This number p is called the dimension of V.

Note: A subspace can have many bases, but all of them are going to consist of same number of vectors = dimension of V.

This theorem follows from property (2) from page 5 above.

Meaning – if $p < q$ then B_2 must be linearly dependent.

Since it is not (B_2 is also a basis) we get $q \leq p$.

Switching the role of $B_2 \longleftrightarrow B_1$ we get $p \leq q$. So $p = q$.

(15.4) Three methods for finding basis.

So far we have seen 2 methods of finding basis of a subspace $V \subset \mathbb{R}^N$, which is given to us as (type II example) spanned by $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$

$$V = \left\{ x_1\vec{v}_1 + x_2\vec{v}_2 + \ldots + x_m\vec{v}_m \text{ where } x_1, \ldots, x_m \in \mathbb{R} \text{ are arbitrary} \right\}$$

Method I. – See example on page 7 of Lecture 14.

Method II. – See Example (ii) from page 3 above.

Method III. – Use the following theorem (proof next time).

$\boxed{\text{Row operations do not change the span of rows of a matrix.}}$

- put $\vec{V}_1, \dots, \vec{V}_m$ as <u>rows</u> to get $A = \begin{bmatrix} - V_1^T - \\ - V_2^T - \\ \vdots \\ - V_m^T - \end{bmatrix}_{m \times N}$
- $A \xrightarrow{\text{Gauss-Jordan}} B = RE(A)$ reduced echelon
- <u>Non-zero rows of B</u> is a basis of V.

Example. Let $\vec{V}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$, $\vec{V}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{V}_3 = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 5 \end{bmatrix}$, $\vec{V}_4 = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -1 \end{bmatrix}$ and

$\vec{V}_5 = \begin{bmatrix} 2 \\ 5 \\ 0 \\ 2 \end{bmatrix}$ be 5 vectors in $\mathbb{R}^4$. Find a basis of the span of $\vec{V}_1, \vec{V}_2, \vec{V}_3, \vec{V}_4$ and $\vec{V}_5$ (say V)

Soln. Put these vectors in <u>rows</u> of a matrix.

$$A = \begin{bmatrix} 1 & 1 & 2 & -1 \\ 1 & 2 & 1 & 1 \\ 1 & 4 & -1 & 5 \\ 1 & 0 & 4 & -1 \\ 2 & 5 & 0 & 2 \end{bmatrix} \xrightarrow[\text{left as an exercise}]{\text{Gauss-Jordan}} \begin{bmatrix} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We get $\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -9 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 4 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ – a basis of V

$\dim(V) = 3$. (Next time – why this method works).