# Problem Set 1

1. Compute $Re(z)$ and $Im(z)$ for $z = \frac{1}{i}, \frac{2}{3-i}, \frac{1-i}{1+i}, (1+i\sqrt{3})^6, (2+i)^3$.

2. Compute $|z|$ and $\arg(z) \in (-\pi, \pi]$ for $z = 3i, -2, 1+i, -1-i$.

3. Determine all $z \in \mathbb{C}$ for which (i) $z^3 = i$, (ii) $z^6 = -8$, (iii) $z^3 = -2+2i$

4. Solve for $z \in \mathbb{C}$: (i) $z^2 + iz + 1 = 0$ (ii) $z^2 - (1+i)z + i = 0$

5. Determine for which complex numbers $z_1, z_2$, the following inequalities are actual equalities.
   (i) $|z_1 + z_2| \leq |z_1| + |z_2|$
   (ii) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

6. For $z_1, z_2 \in \mathbb{C}$, show that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$

7. Let $n \in \mathbb{Z}$, $n \geq 2$ and let $z_1, z_2, \dots, z_n, w_1, w_2, \dots, w_n \in \mathbb{C}$. Prove that:

$$\left|\sum_{k=1}^{n} z_k \overline{w_k}\right|^2 = \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right) - \sum_{1 \leq k < l \leq n} |z_k w_l - z_l w_k|^2$$

(This is called Lagrange's identity. It immediately implies

$$\left|\sum_{k=1}^{n} z_k \overline{w_k}\right|^2 \leq \left(\sum_{k=1}^{n} |z_k|^2\right)\left(\sum_{k=1}^{n} |w_k|^2\right)$$

- Cauchy-Schwarz inequality)

8. Let $z_1, z_2, z_3 \in \mathbb{C}$ be such that $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$. Prove that $z_1, z_2, z_3$ form vertices of an equilateral triangle.

9. Let $z \in \mathbb{C}$ be such that $z^n = 1$ ($n \geq 2$ is an integer). Show that

$$1 + z + z^2 + \dots + z^{n-1} = \begin{cases} n & \text{if } z = 1 \\ 0 & \text{if } z \neq 1 \end{cases}.$$

10. Compute the sums: ($n \in \mathbb{Z}$, $n \geq 2$)
    (i) $\sin(x) + \sin(2x) + \sin(3x) + \dots + \sin((n-1)x)$
    (ii) $\cos(x) + \cos(3x) + \cos(5x) + \dots + \cos((2n-1)x)$

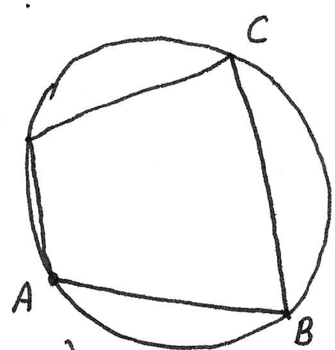
11. Prove the following identities: (i) $\sin(2z-2w) = 2\sin(z-w)\cos(z+w)$

(ii) $\cos(2z) = \cos^2(z) - \sin^2(z)$ (here $z, w \in \mathbb{C}$.)

(iii) $(1-\cos(z-w))(1-\cos(z+w)) = (\cos(z)-\cos(w))^2$

12. Let $z_1 = e^{2i\theta_1}$ and $z_2 = e^{2i\theta_2}$ be two points on the unit circle (i.e., $\theta_1, \theta_2 \in \mathbb{R}$).
Show that $|z_1 - z_2| = 2|\sin(\theta_1 - \theta_2)|$.

13. (Ptolemy Relation) Let ABCD be a quadrilateral inscribed in a circle (as shown in the figure).




Prove that $|AC| \cdot |BD| = |AB| \cdot |CD| + |AD| \cdot |BC|$

(here $|PQ|$ = length of the line segment joining P & Q.)

14. Let $\alpha \in \mathbb{C}$. Prove that there exists $z \in \mathbb{C}$ such that $\sin(z) = \alpha$.

15. Express $\mathrm{Re}(\sin(z))$ and $\mathrm{Im}(\sin(z))$ as functions of $x = \mathrm{Re}(z)$ & $y = \mathrm{Im}(z)$.

16. Sketch the following subsets of $\mathbb{C}$.

(i) ($z_0 \in \mathbb{C}$, $R \in \mathbb{R}_{>0}$ fixed) $\{z \in \mathbb{C} : |z - z_0| \geq R\}$

(ii) $\{z \in \mathbb{C} : |z-2| + |z+2| = 5\}$

(iii) $\{z \in \mathbb{C} : |z-2| - |z+2| > 3\}$

(iv) $\{z \in \mathbb{C} : |z-\alpha| = |z-\beta|\}$ ; $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$.

(v) $\{z \in \mathbb{C} : 0 < \mathrm{Re}(iz) < 1\}$

(vi) $\{z \in \mathbb{C} : |z-1| = \mathrm{Re}(z)\}$

(vii) $-\pi < \theta_1 < \theta_2 < \pi$. $\{z \in \mathbb{C} : \theta_1 < \arg(z) < \theta_2\}$

(viii) $\alpha, \beta \in \mathbb{C}$; $\alpha \neq \beta$. $\{z \in \mathbb{C} : \mathrm{Im}\left(\frac{z-\alpha}{z-\beta}\right) = 0\}$ (for $z = \beta$, take $\frac{z-\alpha}{z-\beta}$ to be real)

(ix) $\{z \in \mathbb{C} : \mathrm{Re}(z^2) = 1\}$

17. Determine all values of $z \in \mathbb{C}$ for which (i) $e^{z} = 1+i$ (ii) $e^{2z+1} = 1$.

18. Find all possible values of $(1-i)^{1+i}$.

19. Use the standard determination of log to compute $\left(\frac{e}{2}(-1-\sqrt{3}\,i)\right)^{3\pi i}$.

20. Let $z \in \mathbb{C}$ be such that $|z| = 1$. Show that all possible values of $z^{i}$ lie on $\mathbb{R}$.

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21. (Circle of Apollonius). Let $z_1, z_2 \in \mathbb{C}$ and $\lambda \in \mathbb{R}_{>0}$ $(\lambda \neq 1)$.

Show that $\left|\dfrac{z - z_1}{z - z_2}\right| = \lambda$ defines a circle.

22. (Bernoulli's lemniscate) $|z^2 - 1| = 1$. Show that $|z|^2 = 2\cos(2\arg(z))$

i.e. the curve given (in polar form) by $r^2 = 2\cos(2\theta)$ – called a lemnisate. What happens if we set $|z^2-1| = \lambda$ $(\lambda \in \mathbb{R}_{>0})$ and vary $\lambda$?

23. (Inverse geometry). Consider the transformation $I(z) = \dfrac{z}{|z|^2}$.

Show that a straight line $L$ not passing through $0$, meeting the unit circle $\{|z|=1\}$ at $z_1$ and $z_2$ is transformed, via $I$, to a circle passing through $0$, $z_1$ and $z_2$.